Forum Geometricorum
Volume 17 (2017) 425–432.

FORUM GEOM
ISSN 1534-1178

Isogonal Conjugates in an n-Simplex

Jawad Sadek

Abstract. As "one of the crown jewels of modern geometry", the symmedian point is not well known by mathematicians or mathematics educators. Locating a person in space without a GPS, or analysing data for a harbor mine-detecting algorithm are but some of the more recent applications to the symmedian point. In this article, we extend the notions of the symmedian point and isogonal conjugates to n-simplexes. Unlike another attempt, our generalisation preserves all the interesting properties of the symmedian point of a triangle. Also, its connection to the least squares point of a simplex is shown. In addition, the symmedian point is given in terms of a simple formula in barycentric coordinates.

1. Introduction

The *symmedian point* of a triangle is one of 11809 known points associated with the geometry of a triangle (see [4] where it is the sixth point on the list). One of the most important properties of the symmedian point is that it coincides with the point at which the sum of the squares of the perpendicular distances from the three sides of the triangle is minimum (the *least squares point*, LSP) [1]. For interesting practical applications to the symmedian point, see [5] and [10]. In [5], the symmedian point shows up in connection with locating a person in space without GPS, and in [10] the symmedian point comes in when analysing data from a test of harbor mine-detecting algorithm. Another interesting property of the symmedian point of a triangle is that, as shown in Figure 1 (b), if the triangle $A'B'C'$ is the pedal triangle of K (i.e., the triangle obtained by projecting K onto the sides of the original triangle), then the symmedian point of the triangle ABC and the centroid of the triangle $A'B'C'$ are concurrent.

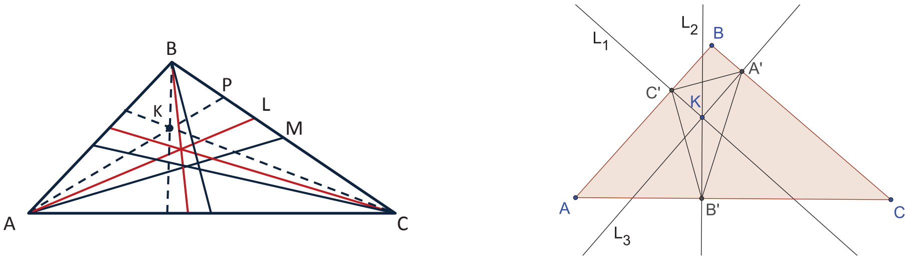
The symmedian point is defined using the concept of isogonal lines. Two lines AR and AS through the vertex A of an angle are said to be isogonal if they are equally inclined from the sides that form $\angle A$. The lines that are isogonal to the medians of a triangle are called symmedian lines [3], pp. 75-76. Figure 1 (a) shows that the symmedian line through $\overline{AP}$ of the triangle ABC is obtained by reflecting the median $\overline{AM}$ through the corresponding angle bisector $\overline{AL}$. The symmedian lines intersect at a single point K known as the symmedian point, also called the Lemoine point.

Publication Date: November 30, 2017. Communicating Editor: Paul Yiu.

The author would like to thank Christina Judd for valuable time and discussion which greatly improved the paper.

The existence of symmedian point of a triangle was proved by the French mathematician Emile Lemoine in 1873 ([3], Chapter 7). Later the symmedian point was defined by Marr for equiharmonic tetrahedrons in 1919 [7]. In [9], an alternate definition of the symmedian point for an *arbitrary* tetrahedron was given. Under the alternate definition, all the properties of the symmedian point for a triangle generalize naturally to a tetrahedron. In the present work we provide the definition and prove the existence of the symmedian point of an arbitrary n-simplex. Then we show that the symmedian point of a simplex coincides with the LSP of that simplex and the centroid of the corresponding pedal simplex. As it was shown in [6], a *least squares solution* method can be used to give an explicit formula for the location of the least squares point and hence the symmedian point in S_n.

The rest of this paper is organized as follows. In section 2, the existence of symmedian point of an n-simplex is proved. In section 3, it is shown that the symmedian point and LSP of an n-simplex are concurrent. In section 4, the concurrency of the symmedian point and the centroid of the corresponding pedal simplex is proved.



(a) (b)

Figure 1. *(a) The symmedian lines of triangle ABC intersect at the symmedian point K. (b) The symmedian point K of the ABC triangle coincides with the centroid $\widehat{C}$ of its pedal triangle, which is the $A'B'C'$ triangle formed by connecting the intersection points of the perpendicular lines L_1, L_2 and L_3 from K to the sides of the ABC triangle.*

2. Symmedian point of an n-simplex and barycentric coordinates

Let $v_0, v_1, \cdots v_n$ be the vertices of an n-simplex S_n in R^n, $n \geq 2$. Recall that a facet of S_n is an $(n-1)$-simplex formed by $(n-1)$ vertices of S_n. Recall also that the angle between two hyperplanes H_i and H_j, $\angle(H_i, H_j)$, is the supplement of the angle between their unit outer normal vectors. Denote by F_i the *facet* formed by the vertices in $\{v_l : l \neq i\}$. Let X and X^* be two points in S_n. Denote by (H_{ij}) and (H^*_{ij}) the hyperplanes formed by joining X and X^* respectively to v_l, $l \neq i, j$. The $(n{-}1)$-dimensional hyperplanes (H_{ij}) and (H^*_{ij}) are said to be *isogonal* conjugates if they are equally inclined from the facets F_i and F_j. (H_{ij}) is called the isogonal conjugate of (H^*_{ij}) and vice versa. Notice that by joining the point X to $(n-2)$ vertices of S_n, $\binom{n+1}{n-1} = \frac{n(n+1)}{2}$ different $(n-1)$-dimensional hyperplanes can be formed, and so there are $\frac{n(n+1)}{2}$ corresponding isogonal conjugate hyperplanes. There is no immediately obvious reason why these $\frac{n(n+1)}{2}$ conjugates should be concurrent. However, that this is always the case will follow from Lemma 2 below. Let M_{ij} be the midpoint of $\overline{v_i v_j}$. The $(n-1)$-hyperplane obtained from M_{ij} and $\{v_l : 0 \leq l \leq n, l \neq i,j\}$ will be called a *med-hyperplane* $(M_{ij} \cup H_{ij})$. The isogonal conjugate to a med-hyperplane $(M_{ij} \cup H_{ij})$ with respect to facets F_i and F_j is called a *symmedian hyperplane*. Taking the midpoints of the $\frac{n(n+1)}{2}$ sides of the simplex S_n, the following lemma shows all med-hyperplanes are concurrent at the centroid G_n of S_n [8].

Lemma 1. *All $\frac{n(n+1)}{2}$ med-hyperplanes planes $(M_{ij} \cup H_{ij})$; $0 \leq i,j, \leq n$, $i \neq j$ are concurrent.*

Proof. Let M_{ij} be the midpoint of the edge $\overline{v_i v_j}$ and $(M_{ij} \cup H_{ij})$ the corresponding med-hyperplane. We show that all med-hyperplanes pass through the centroid G_n of the of S_n given by $G_n = \frac{\sum_0^n v_l}{n+1}$ (see Figure 2 (a)). If $0 \leq k \leq n$, then G_n is located on the segment from an arbitrary vertex v_k to the centroid of the $(n-1)$-simplex $S_{n-1}^{(k)}$ obtained from the vertices in $\{v_l : 0 \leq l \leq n, l \neq k\}$. In fact, let $G_{n-1}^{(k)}$ be the centroid of $S_{n-1}^{(k)}$. Then

$$\begin{aligned} G_n &= \frac{v_0 + \cdots + v_n}{n+1} \\ &= \frac{nG_{n-1}^{(k)}}{n+1} + \frac{v_k}{n+1} \\ &= \alpha G_{n-1}^{(k)} + \beta v_k, \end{aligned}$$

where $\alpha + \beta = 1$. Thus G_n is the on the segment $\overline{G_{n-1}^{(k)} v_k}$. Since each med-hyperplane contains $\overline{G_{n-1}^{(k)} v_k}$ for some k, the result follows.□

Lemma 2. *Consider the n-simplex S_n. Let $X \neq v_i, v_j$ be a point on the hyperplane (H_{ij}) formed by joining X to the vertices in $V_{ij} = \{v_l : 0 \leq l \leq n, l \neq i,j\}$. Let X^* be a point on the isogonal conjugate hyperplane (H_{ij}^*).*

(i): *If $\overline{XP_1}, \overline{XP_2}, \overline{X^*P_2^*}, \overline{X^*P_2^*}$, are the perpendiculars from X and X^* onto the facets F_j and F_i, respectively, then*

$$\frac{XP_1}{XP_2} = \frac{X^*P_2^*}{X^*P_1^*} \tag{1}$$

(ii): *If X is in (H_{ij}) and $\frac{XP_1}{XP_2} = \frac{X^*P_2^*}{X^*P_1^*}$, then X^* is in (H_{ij}^*), where (H_{ij}) and (H_{ij}^*) are isogonal conjugates.*

Proof. (i) We will show the triangles XP_1P_2 and $X^*P_1^*P_2^*$ are similar (see Figure 2 (b)). Let P_3 be the orthogonal projection of P_1 onto the $(n-2)$-hyperplane V_{ij} formed by the vertices $\{v_l : 0 \leq l \leq n, l \neq i,j\}$. Since $\overline{XP_1}$ is orthogonal to F_j, $\overline{XP_1}$ is orthogonal to V_{ij} which is contained in F_j. Thus, the plane formed by the points X, P_1, P_2 is orthogonal to V_{ij}. Similarly, the plane formed by X, P_2, P_3 is orthogonal to V_{ij}. This implies that the points X, P_1, P_2, P_3 are in the same two-dimensional plane. Furthermore, since $\overline{XP_3}$ is orthogonal to V_{ij} and $\overline{XP_1}$ is orthogonal to F_j, $\angle(H_{ij}, F_j) = \angle(\overline{P_1P_3}, \overline{XP_3}) = \angle(\overline{P_1P_2}, \overline{XP_2})$ because the quadrilateral $P_2XP_1P_3$ is cyclic (its vertices are on the same circle). A similar argument yields $\angle(H^*_{ij}, F_i) = \angle(\overline{X^*P^*_1}, \overline{P^*_1P^*_3})$, where P^*_3 is the projection of P^*_1 onto V_{ij}. Since (H_{ij}) and (H^*_{ij}) are isogonal conjugates, $\angle(H^*_{ij}, F_i) = \angle(H_{ij}, F_j)$, and so $\angle(\overline{X^*P^*_1}, \overline{P^*_1P^*_2}) = \angle(\overline{XP_2}, \overline{P_1P_2})$. The similarity of the triangles now follows.

(ii) Since $\frac{XP_1}{XP_2} = \frac{X^*P_2*}{X^*P_1*}$ and $\angle(\overline{XP_1}, \overline{XP_2}) = \angle(\overline{X^*P_1}, \overline{X^*P^*_2})$, the triangles XP_1P_2 and $X^*P_1 * P^*_2$ are similar. It follows that $\angle(F_j, H_{ij}) = \angle(\overline{P_1P_3}, \overline{P_3X}) = \angle(\overline{P_1P_2}, \overline{P_2X}) = \angle(\overline{X^*P^*_1}, \overline{P^*_1P^*_2}) = \angle(\overline{X^*P^*_3}, \overline{P_3P^*_2}) = \angle(F_i, H^*_{ij})$, and so X^* is in the isogonal conjugate of (H_{ij}) □

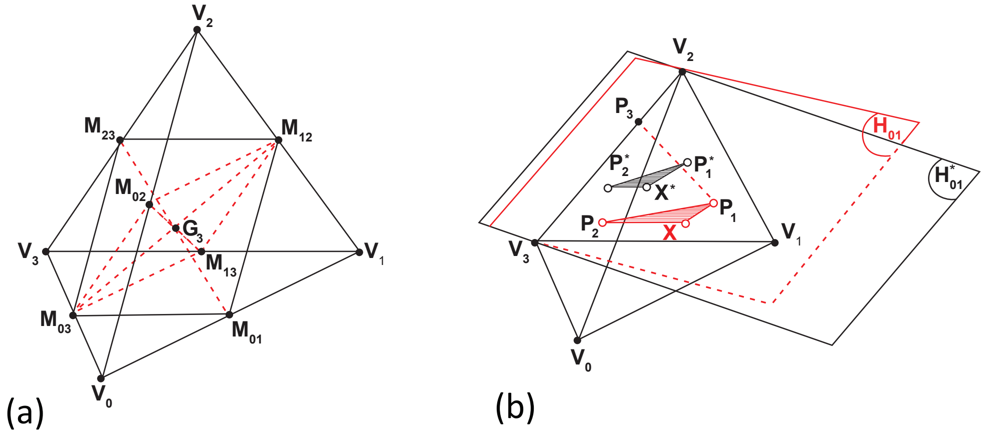


Figure 2. *(a) All six median planes obtained from a side of the 3-simplex* $v_0v_1v_2v_3$ *and the midpoint* M_{ij} *of its opposite side are concurrent. (b) A representation of the isogonal planes* (H_{12}) *and* (H^*_{12})*, and the perpendicular lines from* X *and* X^* *to the facets* $v_1v_2v_3$ *and* $v_0v_2v_3$*, respectively.*

Now we are ready to show the existence of the symmedian point of an n-simplex.

Theorem 3. *Let* (H_{ij}) *be the* $\frac{n(n+1)}{2}$ $(n-1)$*-hyperplanes constructed by joining a point* X *inside the simplex* S_n *to the vertices in* $V_{ij} = \{v_l : 0 \le l \le n, l \ne i, j\}$*. Then the corresponding* $\frac{n(n+1)}{2}$ *isogonal conjugate hyperplanes* (H^*_{ij}) *intersect at a point* X^* *inside* S_n*.*

Proof. Let X^* be the intersection of n isogonal conjugate hyperplanes $(H^*_{i_lj_l})$, $0 \le l \le n-1$, $i_l \ne j_l$. Let $(H_{i_kj_k})$, $k \notin \{0, 1, \cdots, n-1\}$ be a hyperplane passing through X. Let $(H^*_{i_kj_k})$ be the isogonal conjugate to $(H_{i_kj_k})$ with respect to some facets F^*_k and F_k, say. It suffices to show that $X^* \in (H^*_{i_kj_k})$. Let P_1 and P_2 be the orthogonal projections from X to F_k and F^*_k, and let P^*_1 and P^*_2 be the orthogonal projections from X^* to F_k and F^*_k, respectively. In view of Lemma 2 (ii), we need only show $\frac{X^*P^*_1}{X^*P^*_2} = \frac{XP_2}{XP_1}$. In fact, we have

$$\frac{X^*P^*_1}{X^*Q^*} = \frac{XQ}{XP_1}$$

and

$$\frac{X^*Q^*}{X^*P^*_2} = \frac{XP_2}{XQ},$$

where Q and Q^* are the orthogonal projections from X, and X^* onto a facet F_q, $q \leq l \leq n-1$. Multiplying the two left-hand sides and the two right-hand sides of the equalities yields

$$\begin{aligned} \frac{X^*P^*_1}{X^*P^*_2} &= \frac{X^*P^*_1}{X^*Q^*} \times \frac{X^*Q^*}{X^*P^*_2} \\ &= \frac{XQ}{XP_1} \times \frac{XP_2}{XQ} \\ &= \frac{XP_2}{XP_1}. \end{aligned}$$

The desired now follows. Similar argument shows that all the other isogonal conjugate hyperplanes pass through X^*.□
In view of Lemmas 1 and 2, the following definition becomes well-defined.

Definition. The symmedian point of an n-simplex is the isogonal conjugate of the intersection point of all its med-hyperplanes.

Now we briefly explore the relationship between *the barycentric coordinates* of a point X and its isogonal conjugate X^*. Recall that in general, if $x_0, \cdots, x_n$ are the vertices of a simplex S_n in an affine space $\mathcal{A}$ and if $(a_0 + \cdots + a_n)X = a_0x_0 + \cdots + a_nx_n$ and at least one of the $a_i^{,}s$ does not vanish, then we say that the coefficients $(a_0 : \cdots : a_n)$ are *barycentric coordinates* of X, where $X \in \mathcal{A}$ [11]. The barycentric coordinates are homogeneous:

$$(a_0, \cdots, a_n) = (\mu a_0 : \cdots : \mu a_n) \quad \mu \neq 0.$$

Analogous to the triangle case [2], and to the tetrahedron case [9], we have the following property for an n-simplex. Let X be a point in the nth dimensional Euclidean space E^n. Joining X to any n vertices, $n+1$ n-simplexes (H_i), $0 \leq i \leq n$ can be constructed. Let $X = (a_0 : a_2 : \cdots : a_n)$ and $X' = (a'_0 : a'_2 : \cdots : a'_n)$ be the barycentric coordinates of X and X^*, respectively, with respect to S_n. Since *the volumes of these simplexes are proportional to the barycentric coordinates of* X, using lemma 2, we obtain

$$\frac{a'_0 a_0}{(\delta H_0)^2} = \frac{a'_1 a_1}{(\delta H_1)^2} = \cdots = \frac{a'_n a_n}{(\delta H_n)^2} = \mu,$$

where δH_i denote the volume of H_i (its content). It follows that

$$\begin{aligned} X' &= (a'_0 : a'_1 : \cdots : a'_n) \\ &= \left(\mu\frac{(\delta H_0)^2}{a_0} : \mu\frac{(\delta H_1)^2}{a_1} : \cdots : \mu\frac{(\delta H_n)^2}{a_n}\right) \\ &= \left(\frac{(\delta H_0)^2}{a_0} : \frac{(\delta H_1)^2}{a_1} : \cdots : \frac{(\delta H_n)^2}{a_n}\right). \end{aligned} \tag{2}$$

(2) gives an extension of isogonal conjugates to simplexes with a simple formula in barycentric coordinates. Applying (2) to the centroid $(1 : 1 : \cdots : 1)$, we obtain the symmedian point $\Big((\delta H_0)^2 : (\delta H_1)^2 : \cdots : (\delta H_n)^2\Big)$.

3. Concurrency of the Symmedian Point and the Least Squares Point

The least squares point LSP of a given simplex S_n is the point from which the sum of the squares of the perpendicular distances to the facets of S_n is minimized. We show that the symmedian point and the LSP of S_n are concurrent. We start with the following lemma.

Lemma 4. *For an n-simplex S_n, let M_{ij} be the midpoint of $\overline{v_i v_j}$ and $\overline{M_{ij}P}$ and $\overline{M_{ij}Q}$ be the perpendicular line segments from M_{ij} to F_i and F_j, respectively. Then*

$$\frac{M_{ij}Q}{M_{ij}P} = \frac{\delta(F_i)}{\delta(F_j)}, \tag{3}$$

where $\delta(F_k)$ represents the volume (or content) of F_k.

Proof. It is known (and easy to verify using a simple integral) that the volume of S_n is $\frac{1}{n!}$ times the"heights" of vertices, taken in any linear sequence, above the subspace containing the previous vertices [12]. Precisely, the volume $\delta(S_n)$ is given by

$$\delta(S_n) = \frac{1}{n!} h_n h_{n-1} \cdots h_1,$$

where h_1 is simply the distance between the first two vertices, h_2 is the height of the third vertex above the line containing those two vertices, h_3 is the height of the fourth vertex above the plane containing the first three vertices, and so on. Thus $\delta(M_{ij} \cup F_j) = \delta(M_{ij} \cup F_i)$, where $M_{ij} \cup F_k$ is the n-simplex constructed by joining M_{ij} to the vertices of F_k. Let $\overline{v_k H}$ be the perpendicular from v_k to F_k,

$k \neq i, j$. Then

$$\begin{aligned}\frac{1}{n} M_{ij} Q \times \delta(F_j) &= \delta(M_{ij} \cup F_j) \\ &= \delta(M_{ij} \cup F_i) \\ &= \frac{1}{n} M_{ij} P \times \delta(F_i).\end{aligned}$$

This implies (3). □

We're ready to prove the concurrency of the symmedian point and the LSP of S_n.

Theorem 5. *The symmedian point K of S_n coincides with its LSP.*

Proof. First, Lemma 2 (i) together with Lemma 4 imply

$$\frac{x_0}{\delta(F_0)} = \frac{x_1}{\delta(F_1)} = \cdots = \frac{x_n}{\delta(F_n)}, \tag{4}$$

where x_i is the distance from the symmedian point to F_i. If we denote $\delta(F_i)$ by d_i, then Lagrange's identity yields

$$\left(\sum_{i=0}^{n} x_i^2\right)\left(\sum_{i=0}^{n} d_i^2\right) - \left(\sum_{i=0}^{n} x_i d_i\right)^2 = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} (d_i x_j - d_j x_i)^2. \tag{5}$$

Since the d_i's are constant, and $\sum_{i=0}^{n} x_i d_i = n\delta(S_n)$, $\sum_{i=0}^{n} x_i^2$ is minimum if and only if the right hand side of (5) is zero. This occurs only when $x_i d_j = x_j d_i$. In view of (4), this occurs at the symmedian point K, and so the symmedian point coincides with the LSP. □

4. Concurrency of the Symmedian Point and the Centroid of the Corresponding Pedal n-simplex

In this section we show that the symmedian point of an n-simplex S_n coincides with the centroid of the corresponding pedal n-simplex Π_n.

Theorem 6. *The symmedian point of an n-simplex S_n coincides with the centroid of the corresponding pedal n-simplex Π_n.*

Proof. Let K be the symmedian point of S_n. Let the orthogonal projection from K onto a facet F_i intersect F_i at π_i. Let $\widehat{C}$ be the centroid of the pedal simplex Π_n of K. It is well known that $\widehat{C}$ minimizes the sum of the squares of the distances to the vertices $\pi_0, \pi_1, \cdots \pi_n$ of Π_n. Thus,

$$\sum_{i=0}^{n} (\widehat{C}\pi_i)^2 \leq \sum_{i=0}^{n} (X\pi_i)^2 \text{ for any } X \in R^n. \tag{6}$$

Suppose $\widehat{C} \neq K$. Let the orthogonal projections from $\widehat{C}$ to the facet F_i of S_n be c_i. Since K is also the LSP of S_n

$$\sum_{i=0}^{n}(K\pi_i)^2 < \sum_{i=0}^{n}(\widehat{C}c_i)^2. \tag{7}$$

Since c_i is the orthogonal projection from $\widehat{C}$ ontoF_i, $\widehat{C}c_i \leq \widehat{C}\pi_i$ for each i. So using (6) with $X = K$, we have

$$\sum_{i=0}^{n}(\widehat{C}c_i)^2 \leq \sum_{i=0}^{n}(\widehat{C}\pi_i)^2 \leq \sum_{i=0}^{n}(K\pi_i)^2,$$

which contradicts (7). So we must have $\widehat{C} = K$. □

Corollary 7. *The symmedian point and hence the LSP of S_n belongs to its interior.*

Summary. The symmedian point of S_n as defined above is a generalization of the symmedian point for a tetrahedron defined in [9]. This definition differs from the definition of the symmedian point of a tetrahedron given by Marr [7]. Marr's symmedian point of an equiharmonic tetrahedron $ABCD$ (that is, $AD \times BC = AB \times CD = AC \times BD$) is defined as the point of intersection of the lines joining the vertices to the symmedian points of the opposite faces. In [9], an example was given to show that the two symmedian points are in fact different. Our generalization, however, has the advantage of being the same as the least squares point of S_n, as is the case in any triangle. In addition, using a least squares solution technique, the location of the LSP, and hence the symmedian point, of S_n can be given explicitly [6].

References

[1] A. Bogomolny, All About Symmedians, `www.cut-the-knot.org/triangle/symmedians.shtml`.

[2] H. S. M. Coxeter, *Introduction to Geometry*, Wiley, 1963.

[3] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry,* MAA, 1995.

[4] C. Kimberling, *Encyclopedia of Triangle Centers* `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[5] R. Lionheart, Where Am I?, `www.youtube.com/watch?v=pzDyi7ANoVA`

[6] M. Bani-Yaghoub, N. H. Rhee, and J. Sadek, An algebraic method to find the symmedian point of a triangle, *Math. Mag.*, 89 (2016) 197–200.

[7] W. L. Marr, The co-symmedian system of tetrahedra inscribed in a sphere, *Proc. Edinburgh Math. Soc.*, 37 (1918) 59–64.

[8] J. Neuberg, *Memoire sur la tetraedre*, Acad. Royale des Sciences, 37 (1886) 64.

[9] J. Sadek, M. M. Bani-Yaghoub, and N. Rhee, Isogonal conjugates in a tetrahedron, *Forum Geom.*, 16 (2016) 43–50.

[10] R. K. Smither, The symmedian point: constructed and applied, *College Math. Journal*, 42 (2011) 115–117.

[11] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Internat. J. Math. Ed. Sci. Tech.*, 31 (2000) 569–578.

[12] `http://www.mathpages.com/home/kmath664/kmath664.htm`.

Jawad Sadek: Department of Mathematics and Statistics, Northwest Missouri State University, Maryville, Missouri, USA 64468-6001

E-mail address: JAWADS@nwmissouri.edu