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Classical Right-Angled Triangles and the Golden Ratio

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Abstract. In this article, we consider the family of classical right-angle triangles in 2-dimensional Euclidean space. We consider triangle with an arbitrary leg ratio k and show that at $k = 2^p$, where $p = \pm 1$, the area of all built-in triangles is linked to each other by the golden ratio φ. Keeping $k = 2^p$, we address changes in above triangles occurring at the planar similarity transformation, prove an invariancy of the area ratio between predecessor and successor triangle and show that evolution curve is a logarithmic spiral. Reason of such geometrical features is associated with the unique nature of φ providing parity between the linear and non-linear properties of geometry objects.

1. Theorem on area ratio

In 2-dimensional Euclidean space, consider arbitrary right-angled $\triangle ABC$ such that ratio $AC/BC = k$, where k is any rational number (Figure 1).

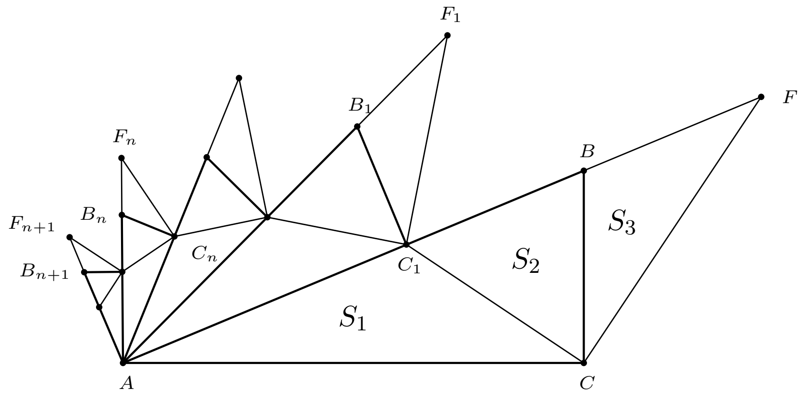

Figure 1. Family of triangles at planar similarity transformation. [1]

Mark off distance from the vertex B as $BC_1 = BC$ and $BF = BC$. Since $AC = k \cdot BC$, $AB = BC\sqrt{k^2+1}$, $AC_1 = AB - BC_1 = BC(\sqrt{k^2+1} - 1)$ and $AF = AB + BC = BC(\sqrt{k^2+1} + 1)$. Denote by S_1 the area of $\triangle AC_1C$, S_2 the area of $\triangle CC_1B$, S_3 the area of $\triangle BCF$, $S_1 + S_2 + S_3 = S$, the golden ratio [1] as φ, and

$$f_k = \frac{AF}{AC} = \frac{AC}{AC_1} = \frac{\sqrt{k^2+1}+1}{k}. \tag{1}$$

Then,

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[1]In Figure 1, successive changes in family of the right-angle triangles at planar similarity transformation are shown. The ordinate is original vertical edge in triangle ABC, while the abscissa is original horizontal edge in triangle ABC. Plot represents family of the similar right-angled triangles in 2-dimensional Euclidean space keeping the leg's ratio $k = 2$.

Theorem 1. *If in right-angled triangle ABC with leg ratio $\frac{AC}{BC} = 2^p$, where $p = \pm 1$, to mark off from the vertex B along hypotenuse AB the line segments BC_1 and BF with the length equal to length of BC, then the ratio*

$$\left(\frac{S}{S_2 + S_3}\right)^p = \left(\frac{S_2 + S_3}{S_1}\right)^p = \left(\frac{AF}{AC}\right)^p = \left(\frac{AC}{AC_1}\right)^p = \varphi^p. \tag{2}$$

First, we prove some lemmas.

Lemma 2. *The areas*

$$S_2 = S_3 = \frac{BC^2}{2}\sqrt{1 - \frac{1}{k^2+1}}. \tag{3}$$

Proof. Since $\cos ABC = -\cos CBF = \frac{1}{\sqrt{k^2+1}}$, for $\triangle CC_1B$, using the law of cosines $C1C^2 = 2BC^2\left(1 - \frac{1}{\sqrt{k^2+1}}\right)$. In the same way, in $\triangle BCF$, $CF^2 = 2BC^2\left(1 + \frac{1}{\sqrt{k^2+1}}\right)$. Then, employing Heron's formula for area [2], we obtain (3). □

Lemma 3. *Area*

$$S_1 = \frac{BC^2}{2}\left(k - \sqrt{1 - \frac{1}{k^2+1}}\right). \tag{4}$$

Proof. The area of $\triangle ABC = k \cdot BC^2/2$. From (3), we obtain (4). □

Lemma 4. *If $k = 2$, the ratio*

$$\frac{S_2 + S_3}{S_1} = \varphi. \tag{5}$$

Proof. From (3) and (4),

$$\begin{aligned}\frac{S_2 + S_3}{S_1} &= 2\sqrt{1 - \frac{1}{k^2+1}}\left(k - \sqrt{1 - \frac{1}{k^2+1}}\right)^{-1} \\ &= \frac{1}{\sqrt{k^2+1} - 1} = \frac{2}{k} \cdot \frac{\sqrt{k^2+1} + 1}{k} = \frac{2}{k} \cdot f_k.\end{aligned} \tag{6}$$

So, at $k = 2$, using (1), $\frac{S_2+S_3}{S_1} = f_k = \frac{\sqrt{5}+1}{2} = \varphi$. □

Lemma 5. *If $k = 2$, the ratio*

$$\frac{S}{S_2 + S_3} = \varphi. \tag{7}$$

Proof. Modify (7) as

$$\frac{S}{S_2 + S_3} = 1 + \frac{S_1}{S_2 + S_3} = f_k. \tag{8}$$

From (5),

$$\frac{S_1}{S_2 + S_3} = \frac{1}{f_k} \cdot \frac{k}{2}.$$

Then (8) becomes

$$1 + \frac{1}{f_k} \cdot \frac{k}{2} = f_k, \tag{9}$$

and at $k = 2$, the positive root of (9) is $f_k = \varphi$.

□

Obviously, all reasoning in Lemmas 4 and 5 is valid for $p = -1$ if we replace $k = 2$ with $k = \frac{1}{2}$, and rename the vertices A and B. Now, we can prove Theorem 1.

Proof of Theorem 1

In $\triangle ABF$, from (1), (5), and (7), immediately follows (2).

Corollary 6.

$$(SS_1)^p = (S_2 + S_3)^{2p}. \tag{10}$$

2. Evolution theorem

For the rest of this article, we will use $k = 2^p$. So, in $\triangle AB_1C_1$, from (2), $AC_1 = \frac{AC}{\varphi^p}$. From the point C_1 raise a perpendicular to AB, then on this perpendicular mark off the line segment B_1C_1 equal to $\frac{BC}{\varphi}$ and connect the points B_1 and A. Since $AB_1 = \frac{AB}{\varphi^p}$, triangle AB_1C_1 is similar to ABC as having three proportional edges and, obviously, the area of $\triangle AB_1C_1$ is $\frac{S}{\varphi^{2p}}$. Continuing this way, at an arbitrary step n, $n > 1$, in triangle AB_nC_n, $\frac{AC}{AC_n} = \varphi^{pn}$, and the area S_n of $\triangle AB_nC_n$ is $\frac{S}{\varphi^{2pn}}$. Further, we will call similar triangles AF_nC_n as the full ones with area S_n^F, $\triangle AC_{n+1}C_n$ as the core ones with area S_n^C in contrast to the transfer triangles, $C_nC_{n+1}B_n$ and $C_nF_nB_n$ with area S_n^T.

Theorem 7. *The full area S_{n+1}^F (S_n^F) of the first immediate successor (precursor) in the chain of similar triangles is exactly equal to the area S_n^C (S_{n+1}^C) of the core triangle in its immediate precursor (successor), i.e.*

$$S_{n+1}^F = S_n^C, \quad (S_n^F = S_{n+1}^C). \tag{11}$$

Proof. For the case $\partial S_n/\partial S < 0$, where S_n is the area of triangle AB_nC_n, n is the number of evolution steps.

$$S_{n+1}^F = \frac{S_n^F}{\varphi^2}. \tag{12}$$

From (7),

$$\frac{(S_n^F)^2}{4(S_n^T)^2} = \varphi^2. \tag{13}$$

Insert (13) into (12) and apply (10) for an arbitrary n, which is $4(S_n^T)^2 = S_n^F S_n^C$, then

$$S_{n+1}^F = 4\frac{S_n^F (S_n^T)^2}{(S_n^F)^2} = 4\frac{(S_n^T)^2}{S_n^F} = \frac{S_n^F S_n^C}{S_n^F} = S_n^C.$$

□

For the case $\partial S_n/\partial n > 0$, using the same logic as above,

$$S_{n+1}^C = \varphi^2 S_n^C = S_n^C \frac{(S_n^F)^2}{4(S_n^T)^2} = S_n^C \frac{(S_n^F)^2}{S_n^F S_n^C} = S_n^F.$$

Theorem 8. *At $\partial S_n/\partial n < 0$, the full area S_{n+1}^F of the first immediate successor in the chain of similar triangles is exactly equal to the area S_n^F of its immediate precursor decremented by the double area S_n^T of transfer triangle, i.e.*

$$S_{n+1}^F = S_n^F - 2S_n^T, \tag{14a}$$

while at $\partial S_n/\partial S > 0$, incremented by the double area S_n^T of transfer triangle multiplied by $\varphi + 1$, i.e.

$$S_{n+1}^F = S_n^F + 2(\varphi + 1)S_n^T. \tag{14b}$$

Proof. For the case $\partial S_n/\partial n < 0$, obviously,

$$S_n^C = S_n^F - 2S_n^T. \tag{15}$$

Then, by Theorem 7, replace in (15) S_n^C with S_{n+1}^F, and obtain (14a).

For the case $\partial S_n/\partial S > 0$, $S_{n+1}^F = S_{n+1}^C + 2S_{n+1}^T$. Since $S_{n+1}^C = S_n^F$ and $S_{n+1}^T = \varphi^2 S_n^T = (\varphi + 1)S_n^T$, we obtain (14b). □

3. Transformation curve

Theorem 9. *At planar similarity transformation, the plane curve dealing the path for the triangle's edge in the chain of similar triangles is a logarithmic spiral.*

Proof. In the case $\partial S_n/\partial n < 0$, the decrement $\Delta AC = \frac{AC}{\varphi} - AC = AC\left(\frac{1}{\varphi} - 1\right)$ occurred at the turn of AC to $\Delta\theta_n = \angle C_n AB_n = \arctan\frac{1}{k}$ (see Figure 1). Going to the differential values $d(AC) = AC\left(\frac{1}{\varphi} - 1\right)d\theta$, we have

$$AC = a \cdot \exp\left(\left(\frac{1}{\varphi} - 1\right)b\theta\right),$$

which is the equation of folding logarithmic spiral with the growth factor $b\left(\frac{1}{\varphi} - 1\right)$, $b = \frac{2\pi}{\arctan\frac{1}{2}}$, and $a = AC_0$.

For the case $\partial S_n/\partial n > 0$, $\Delta AC = AC(\varphi - 1)$ which yields the unfolding logarithmic spiral with the growth factor $b(\varphi - 1)$:

$$AC = a \cdot \exp\left((\varphi - 1)b\theta\right) = a \cdot \exp\left(\frac{b}{\varphi}\theta\right).$$

□

4. Conclusions and outlook

We have shown that in the right-angle triangle with leg's ratio $k = 2^p$, where $p = \pm 1$, the ratio of the edges and areas of built-in triangles is obeyed to the golden ratio relation. At planar similarity transformation at each step, the full area of the successor (predecessor) triangle is exactly equal to the area of the built-in core triangle of its predecessor (successor). It assumes that triangle's transformation takes such the way to preserve the key parameters of predecessor triangle irrelevant to the sign of $\frac{\partial S_n}{\partial n}$, i.e. the ratio $\frac{S^F_{n+1}}{S^C_n}\left(\frac{S^F_n}{S^C_{n+1}}\right) = 1$ and it is invariant under above transformation.

Saying about the directions for future investigations we note that it would be interesting to consider applicability of above results for the three and higher dimension case. So, we look forward to reporting this in our future work.

References

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