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Orthocenters of Simplices in Hyperbolic Spaces

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Abstract. We consider orthocenters of simplices of the hyperbolic space. Unlike the cases of the Euclidean spaces or the sphere, the similar condition does not always imply the existence of the orthocenter. In this paper, we give a characterization of the existence of the orthocenter.

1. Introduction

For a simplex of the Euclidean space, orthocenters exist only at most one point, and the existence of the orthocenter is equivalent to

$$(\mathbf{q}_i - \mathbf{q}_k) \cdot (\mathbf{q}_j - \mathbf{q}_\ell) = 0$$

for arbitrary pairwise distinct vertices $\mathbf{q}_i$, $\mathbf{q}_j$, $\mathbf{q}_k$, $\mathbf{q}_\ell$ (see, e.g., §1 of [2], (1.1) of [4]). For a simplex of the unit sphere of the Euclidean space, the existence of orthocenters is equivalent to

$$(\mathbf{p}_i^* \cdot \mathbf{p}_k^*)(\mathbf{p}_j^* \cdot \mathbf{p}_\ell^*) = (\mathbf{p}_i^* \cdot \mathbf{p}_\ell^*)(\mathbf{p}_j^* \cdot \mathbf{p}_k^*)$$

for arbitrary pairwise distinct vertices $\mathbf{p}_i^*$, $\mathbf{p}_j^*$, $\mathbf{p}_k^*$, $\mathbf{p}_\ell^*$. Moreover, if there exist orthocenters, the uniqueness of the pair of antipodal orthocenters each other is equivalent that there exist at least two pairs of vertices $\{\mathbf{p}_i^*, \mathbf{p}_k^*\}$ and $\{\mathbf{p}_j^*, \mathbf{p}_\ell^*\}$ such that $\mathbf{p}_i^* \cdot \mathbf{p}_k^* \neq 0 \neq \mathbf{p}_j^* \cdot \mathbf{p}_\ell^*$. (Theorems 1 & 2 of [7]). In this paper we consider the orthocenter of the hyperbolic space. For a simplex of the Minkowski model of the hyperbolic space of the Minkowski space, orthocenters exist only at most one point, and if the orthocenter exists, then it holds that

$$\langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_\ell^* \rangle = \langle \mathbf{p}_i^* | \mathbf{p}_\ell^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle \qquad \text{(a)}^*$$

for arbitrary pairwise distinct vertices $\mathbf{p}_i^*$, $\mathbf{p}_j^*$, $\mathbf{p}_k^*$, $\mathbf{p}_\ell^*$, where $\langle \cdot | \cdot \rangle$ means the pseudo-inner product of the Minkowski space. However, unlike two cases above, the condition (a)* does not always imply the existence of the orthocenter. The existence of it is equvalent to (a)* and the positivity of $-\langle \tilde{\mathbf{h}} \,|\, \tilde{\mathbf{h}} \rangle$ (Theorem 21), where $\tilde{\mathbf{h}}$ is a vector whose normalization is the orthocenter of the simplex of Minkowski model of the hyperbolic space (see the formula (73)). In the following figures, dashed curves, solid curves, dotted curves, and the black point are boundaries of Poincaré model of the 2-dimensional hyperbolic space, boundaries of simplices,

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pseudo-perpendiculars from vertices to the opposite faces, and the orthocenter, respectively (for the relation of Minkowski model and Poincaré model, see Remark 1).

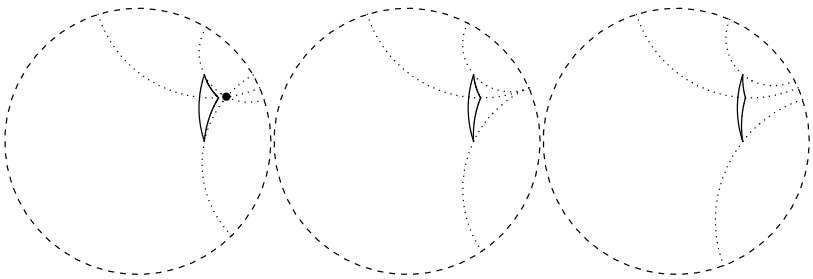

Figure 1.

2. Linear algebra.

We prepare three lemmas in this section. For $\mathbf{a}_0, \ldots, \mathbf{a}_{n-2} \in \mathbb{R}^n$, $\langle\langle \mathbf{a}_0, \ldots, \mathbf{a}_{n-2} \rangle\rangle \in \mathbb{R}^n$ is the unique vector such that

$$\langle\langle \mathbf{a}_0, \ldots, \mathbf{a}_{n-2} \rangle\rangle \cdot \mathbf{a}_{n-1} = \det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \tag{1}$$

for all $\mathbf{a}_{n-1} \in \mathbb{R}^n$, where $(\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}, \mathbf{a}_{n-1})$ is the matrix that has $\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}$, $\mathbf{a}_{n-1}$ as column vectors with Cartesian coordinate system [6]. The first lemma is the following.

Lemma 1. *For* $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1} \in \mathbb{R}^n$, *let*

$$\tilde{\mathbf{a}}_k = (-1)^{n-1-k} \langle\langle \mathbf{a}_0, \ldots \widehat{\mathbf{a}_k} \ldots, \mathbf{a}_{n-1} \rangle\rangle \tag{2}$$

for $k = 0, \ldots, n-1$, *where the circumflex indicates that the term below it has been omitted. Then we have*

$$(\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1})^T = \mathrm{adj}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}), \tag{3}$$

$$\det(\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1}) = (\det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}))^{n-1}, \tag{4}$$

$$(-1)^{n-1-k} \langle\langle \tilde{\mathbf{a}}_0, \ldots, \widehat{\tilde{\mathbf{a}}_k}, \ldots, \tilde{\mathbf{a}}_{n-1} \rangle\rangle = (\det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}))^{n-2} \mathbf{a}_k, \tag{5}$$

where adj $\mathbf{A}$ *is the transpose of the cofactor matrix of* $\mathbf{A}$.

Proof. The equation (3) comes from the first equation of Lemma 3.4 of [6] or from two equations

$$(\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1})^T \cdot (\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) = \det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) \cdot \mathbf{E}_n, \tag{6}$$

$$\mathrm{adj}\, \mathbf{A} \cdot \mathbf{A} = \det \mathbf{A} \cdot \mathbf{E}_n, \tag{7}$$

where (6) comes from (1) and (2), and (7) is an identity of an arbitrary square matrix $\mathbf{A}$ of degree n and the unit matrix $\mathbf{E}_n$ of degree n (notice that (6) and (7) imply (3) even if $\det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) = 0$, because (3) becomes a polynomial identity if elements of matrices are replaced with variables. The same is true for the

following equalities. See Chapter II, §5 of [5]). The equation (4) comes from (3) and the identity

$$\det(\operatorname{adj}\mathbf{A}) = (\det\mathbf{A})^{n-1}, \tag{8}$$

which comes from (7). The equation (3) means the identity of vectors $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}$:

$((-1)^{n-1}\langle\langle\mathbf{a}_1,\ldots,\mathbf{a}_{n-1}\rangle\rangle,\ldots,(-1)^{n-1-k}\langle\langle\mathbf{a}_0,\ldots\widehat{\mathbf{a}_k}\ldots,\mathbf{a}_{n-1}\rangle\rangle,\ldots,\langle\langle\mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\rangle)^T$
$= (\tilde{\mathbf{a}}_0,\ldots,\tilde{\mathbf{a}}_k,\ldots,\tilde{\mathbf{a}}_{n-1})^T = \operatorname{adj}(\mathbf{a}_0,\ldots,\mathbf{a}_{n-1})$.

So we also have the equation replaced $\mathbf{a}_0,\ldots,\mathbf{a}_{n-1}$ with $\tilde{\mathbf{a}}_0,\ldots,\tilde{\mathbf{a}}_{n-1}$:

$((-1)^{n-1}\langle\langle\tilde{\mathbf{a}}_1,\ldots,\tilde{\mathbf{a}}_{n-1}\rangle\rangle,\cdots,(-1)^{n-1-k}\langle\langle\tilde{\mathbf{a}}_0,\ldots\widehat{\tilde{\mathbf{a}}_k}\ldots,\tilde{\mathbf{a}}_{n-1}\rangle\rangle,\cdots,\langle\langle\tilde{\mathbf{a}}_0,\ldots,\tilde{\mathbf{a}}_{n-2}\rangle\rangle)^T$
$= \operatorname{adj}(\tilde{\mathbf{a}}_0,\ldots,\tilde{\mathbf{a}}_{n-1})$,

which implies (5):

$((-1)^{n-1}\langle\langle\tilde{\mathbf{a}}_1,\ldots,\tilde{\mathbf{a}}_{n-1}\rangle\rangle,\cdots,(-1)^{n-1-k}\langle\langle\tilde{\mathbf{a}}_0,\ldots\widehat{\tilde{\mathbf{a}}_k}\ldots,\tilde{\mathbf{a}}_{n-1}\rangle\rangle,\cdots,\langle\langle\tilde{\mathbf{a}}_0,\ldots,\tilde{\mathbf{a}}_{n-2}\rangle\rangle)$
$= (\operatorname{adj}(\tilde{\mathbf{a}}_0,\ldots,\tilde{\mathbf{a}}_{n-1}))^T = \operatorname{adj}((\tilde{\mathbf{a}}_0,\ldots,\tilde{\mathbf{a}}_{n-1})^T)$
$= \operatorname{adj}(\operatorname{adj}(\mathbf{a}_0,\ldots,\mathbf{a}_{n-1})) = (\det(\mathbf{a}_0,\ldots,\mathbf{a}_{n-1}))^{n-2}(\mathbf{a}_0,\ldots,\mathbf{a}_k,\ldots,\mathbf{a}_{n-1})$,

where the third equality comes from (3) and the last equality comes from the identity

$$\operatorname{adj}(\operatorname{adj}\mathbf{A}) = (\det\mathbf{A})^{n-2}\mathbf{A}, \tag{9}$$

which comes from (7), (8), and the identity replaced $\mathbf{A}$ of (7) with $\operatorname{adj}\mathbf{A}$. □

$\langle\cdot\,|\,\cdot\rangle$ is the pseudo-inner product of $\mathbb{R}^n$, that is,

$$\langle\mathbf{x}|\mathbf{y}\rangle = \iota(\mathbf{x})\,.\,\mathbf{y} \in \mathbb{R}, \tag{10}$$

for $\mathbf{x} = (\underline{\mathbf{x}}^T, x)^T$, $\mathbf{y} = (\underline{\mathbf{y}}^T, y)^T \in \mathbb{R}^{n-1}\times\mathbb{R} = \mathbb{R}^n$, where

$$\iota(\mathbf{x}) = (\underline{\mathbf{x}}^T, -x)^T \in \mathbb{R}^n, \tag{11}$$

i.e.,

$$\langle\mathbf{x}|\mathbf{y}\rangle = \langle\mathbf{y}|\mathbf{x}\rangle = \underline{\mathbf{x}}\,.\,\underline{\mathbf{y}} - xy. \tag{12}$$

$\mathbb{R}^n$ with the pseudo-inner product $\langle\cdot\,|\,\cdot\rangle$ is called Minkowski space. The second lemma is the following.

Lemma 2. *Let* $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}, \mathbf{b}_0, \ldots, \mathbf{b}_{n-1} \in \mathbb{R}^n$ *and* $m = 0, \ldots, n-1$. *Then, we have*

$$\iota(\langle\langle\mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\rangle) = -\langle\langle\iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-2})\rangle\rangle; \tag{13}$$

$$\begin{aligned}\det(\mathbf{a}_0,\ldots,\mathbf{a}_{n-1})\det(\mathbf{b}_0,\ldots,\mathbf{b}_{n-1}) &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-1}^{k=0,\ldots,n-1}\\ &= (-1)^{n-1}\det(-\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-1}^{k=0,\ldots,n-1};\end{aligned} \tag{14}$$

$$\begin{aligned}\langle\langle\mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\rangle\,|\,\langle\langle\mathbf{b}_0,\ldots,\mathbf{b}_{n-2}\rangle\rangle &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-2}^{k=0,\ldots,n-2}\\ &= (-1)^{n-2}\det(-\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-2}^{k=0,\ldots,n-2};\end{aligned} \tag{15}$$

if $\mathbf{b}_m \in \{\underline{\mathbf{0}}\}\times\mathbb{R}$,

$$\begin{aligned}\det(\mathbf{a}_k\,.\,\mathbf{b}_\ell)_{\ell=0,\ldots,m}^{k=0,\ldots,m} &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,m}^{k=0,\ldots,m}\\ &= (-1)^m\det(-\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,m}^{k=0,\ldots,m},\end{aligned} \tag{16}$$

where $\underline{\mathbf{0}}$ *is the zero vector in* $\mathbb{R}^{n-1}$*, and we use the following notation:*

$$(F(k,\ell))_{\ell=\ell_0,\ldots,\ell_{h-1}}^{k=k_0,\ldots,k_{h-1}} = \left.\begin{pmatrix} \overbrace{\begin{matrix} \vdots \\ \cdots \quad F(k,\ell) \quad \cdots \\ \vdots \end{matrix}}^{\ell=\ell_0,\ldots,\ell_{h-1}} \end{pmatrix}\right\} k=k_0,\ldots,k_{h-1}.$$

Proof. The latter equalities of (14), (15), and (16) are trivial, so it is enough to show the former equalities of them and (13). For the equation (13), we have

$$\begin{aligned} \langle -\iota(\langle\!\langle \mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\!\rangle)|\mathbf{a}'_{n-1}\rangle &= -\langle\!\langle \mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\!\rangle \cdot \mathbf{a}'_{n-1} \\ &= -\det\left(\mathbf{a}_0,\ldots,\mathbf{a}_{n-2},\mathbf{a}'_{n-1}\right) \\ &= \det\left(\iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-2}),\iota(\mathbf{a}'_{n-1})\right) \\ &= \langle\!\langle \iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-2})\rangle\!\rangle \cdot \iota(\mathbf{a}'_{n-1}) \\ &= \langle\langle\!\langle \iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-2})\rangle\!\rangle|\mathbf{a}'_{n-1}\rangle \end{aligned}$$

where $\mathbf{a}'_{n-1}$ is an arbitrary vector in $\mathbb{R}^n$. For (14), we have

$$\begin{aligned} \det(\mathbf{a}_0,\ldots,\mathbf{a}_{n-1})\det(\mathbf{b}_0,\ldots,\mathbf{b}_{n-1}) &= -\det(\iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-1}))\det(\mathbf{b}_0,\ldots,\mathbf{b}_{n-1}) \\ &= -\det(\iota(\mathbf{a}_k)\cdot\mathbf{b}_\ell)_{\ell=0,\ldots,n-1}^{k=0,\ldots,n-1} \\ &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-1}^{k=0,\ldots,n-1}. \end{aligned}$$

For (15), we have

$$\begin{aligned} \langle\langle\!\langle \mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\!\rangle|\langle\!\langle \mathbf{b}_0,\ldots,\mathbf{b}_{n-2}\rangle\!\rangle\rangle &= \iota(\langle\!\langle \mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\!\rangle)\cdot\langle\!\langle \mathbf{b}_0,\ldots,\mathbf{b}_{n-2}\rangle\!\rangle \\ &= -\langle\!\langle \iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-2})\rangle\!\rangle\cdot\langle\!\langle \mathbf{b}_0,\ldots,\mathbf{b}_{n-2}\rangle\!\rangle \\ &= -\det(\iota(\mathbf{a}_k)\cdot\mathbf{b}_\ell)_{\ell=0,\ldots,n-2}^{k=0,\ldots,n-2} \\ &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-2}^{k=0,\ldots,n-2}, \end{aligned}$$

where the second equality comes from (13) and the third equality comes from Lemma 3.4 of [6]. For (16), we have

$$\begin{aligned} &\det(\mathbf{a}_k\cdot\mathbf{b}_\ell)_{\ell=0,\ldots,m}^{k=0,\ldots,m} \\ &= b_m\det\begin{pmatrix} \underline{\mathbf{a}}_0\cdot\underline{\mathbf{b}}_0+a_0b_0 & \cdots & \underline{\mathbf{a}}_0\cdot\underline{\mathbf{b}}_{m-1}+a_0b_{m-1} & a_0 \\ \vdots & & \vdots & \vdots \\ \underline{\mathbf{a}}_m\cdot\underline{\mathbf{b}}_0+a_mb_0 & \cdots & \underline{\mathbf{a}}_m\cdot\underline{\mathbf{b}}_{m-1}+a_mb_{m-1} & a_m \end{pmatrix} \\ &= b_m\det\begin{pmatrix} \underline{\mathbf{a}}_0\cdot\underline{\mathbf{b}}_0-a_0b_0 & \cdots & \underline{\mathbf{a}}_0\cdot\underline{\mathbf{b}}_{m-1}-a_0b_{m-1} & a_0 \\ \vdots & & \vdots & \vdots \\ \underline{\mathbf{a}}_m\cdot\underline{\mathbf{b}}_0-a_mb_0 & \cdots & \underline{\mathbf{a}}_m\cdot\underline{\mathbf{b}}_{m-1}-a_mb_{m-1} & a_m \end{pmatrix} \\ &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,m}^{k=0,\ldots,m}, \end{aligned}$$

where $\mathbf{a}_k = ({\underline{\mathbf{a}}_k}^T, a_k)^T$ for $k = 0,\ldots, m$, $\mathbf{b}_\ell = ({\underline{\mathbf{b}}_\ell}^T, b_\ell)^T$ for $\ell = 0,\ldots, m-1$, and $\mathbf{b}_m = (\underline{\mathbf{0}}^T, b_m)^T$. □

The third lemma is the Cauchy-Schwarz inequality on Minkowski space.

Lemma 3. *For* $\mathbf{a} = (\underline{\mathbf{a}}^T, a)^T$, $\mathbf{b} = (\underline{\mathbf{b}}^T, b)^T \in \mathbb{R}^n$, *if* $|\underline{\mathbf{a}}| \leqq a$ *and* $|\underline{\mathbf{b}}| \leqq b$, *then we have*

$$-\langle \mathbf{a}|\mathbf{b}\rangle \geqq \sqrt{-\langle \mathbf{a}|\mathbf{a}\rangle}\sqrt{-\langle \mathbf{b}|\mathbf{b}\rangle}. \tag{17}$$

Moreover, if $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b}$ *and* $\mathbf{a}$ *and* $\mathbf{b}$ *are not parallel, i.e., if there does not exist* $c \in \mathbb{R} \setminus \{0\}$ *such that* $c\mathbf{a} = \mathbf{b}$, *then we also have*

$$-\langle \mathbf{a}|\mathbf{b}\rangle > \sqrt{-\langle \mathbf{a}|\mathbf{a}\rangle}\sqrt{-\langle \mathbf{b}|\mathbf{b}\rangle}. \tag{18}$$

Proof. We have

$$ab - \underline{\mathbf{a}} \,.\, \underline{\mathbf{b}} \geqq ab - |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \geqq \sqrt{a^2 - |\underline{\mathbf{a}}|^2}\sqrt{b^2 - |\underline{\mathbf{b}}|^2}, \tag{19}$$

where the former and latter inequalities come from $\underline{\mathbf{a}} \,.\, \underline{\mathbf{b}} \leqq |\underline{\mathbf{a}}||\underline{\mathbf{b}}|$, and

$$(ab - |\underline{\mathbf{a}}||\underline{\mathbf{b}}|)^2 - (a^2 - |\underline{\mathbf{a}}|^2)(b^2 - |\underline{\mathbf{b}}|^2) = (b|\mathbf{a}| - a|\mathbf{b}|)^2 \geqq 0,$$

respectively. Moreover, if $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b}$ (which means $a > 0$ and $b > 0$) and $\mathbf{a}$ and $\mathbf{b}$ are not parallel, one of two inequalities "$\geqq$" of (19) can be replaced with "$>$", because $\frac{b}{a}\underline{\mathbf{a}} \neq \underline{\mathbf{b}}$ implies either $\underline{\mathbf{a}} \,.\, \underline{\mathbf{b}} < |\underline{\mathbf{a}}||\underline{\mathbf{b}}|$ or $b|\underline{\mathbf{a}}| \neq a|\underline{\mathbf{b}}|$. In particular, if

$$\underline{\mathbf{a}} \,.\, \underline{\mathbf{b}} = |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \tag{20}$$

and

$$b|\underline{\mathbf{a}}| = a|\underline{\mathbf{b}}|, \tag{21}$$

then we have $\underline{\mathbf{a}} = \underline{\mathbf{0}} = \underline{\mathbf{b}}$ or $\underline{\mathbf{a}} \neq \underline{\mathbf{0}} \neq \underline{\mathbf{b}}$. In the former case, we have $\frac{b}{a}\mathbf{a} = \frac{b}{a}(\underline{\mathbf{0}}^T, a)^T = (\underline{\mathbf{0}}^T, b)^T = \mathbf{b}$, which is a contradiction. In the latter case, there exists $c > 0$ such that $c\underline{\mathbf{a}} = \underline{\mathbf{b}}$ from (20) and we also have $c = \frac{b}{a}$ from (21), which is a contradiction. □

Corollary 4. *For* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, *if* $\langle \mathbf{a}|\mathbf{a}\rangle < 0$ *and* $\langle \mathbf{a}|\mathbf{b}\rangle = 0$ *then* $\langle \mathbf{b}|\mathbf{b}\rangle > 0$.

Proof. It is enough to show that $\langle \mathbf{a}|\mathbf{a}\rangle < 0$ and $\langle \mathbf{b}|\mathbf{b}\rangle \leqq 0$ imply $\langle \mathbf{a}|\mathbf{b}\rangle \neq 0$. If there exists $c \in \mathbb{R} \setminus \{0\}$ such that $c\mathbf{a} = \mathbf{b}$, we have $\langle \mathbf{a}\,|\,\mathbf{b}\rangle = c\langle \mathbf{a}\,|\,\mathbf{a}\rangle \neq 0$. Otherwise, from Lemma 3, we have one of the following four inequalities: $-\langle \mathbf{a}\,|\,\mathbf{b}\rangle > 0$, $-\langle \mathbf{a}|-\mathbf{b}\rangle > 0$, $-\langle -\mathbf{a}|\mathbf{b}\rangle > 0$, or $-\langle -\mathbf{a}|-\mathbf{b}\rangle > 0$. □

3. Hyperbolic simplices.

Let

$$\mathbb{H}^{n-1} = \{\mathbf{x} = (\underline{\mathbf{x}}^T, x)^T \in \mathbb{R}^n : \langle \mathbf{x}|\mathbf{x}\rangle = -1, x > 0\},$$

which is Minkowski model of the hyperbolic space with constant curvature -1 (for hyperbolic geometry, see e.g., [1] or [8]).

Remark 1. The following figure is the relation of Minkowski model and Poincaré model, $\mathrm{int}(2 \cdot \mathbb{D}^{n-1}) = \{\underline{\mathbf{y}} \in \mathbb{R}^{n-1} : |\underline{\mathbf{y}}| < 2\}$, of the hyperbolic space (the correspondence of two models is $\underline{\mathbf{y}} = 2\underline{\mathbf{x}}/(x+1)$).

In hyperbolic space, an arbitrary point can be moved to another arbitrary point by a transformation preserving the pseudo-inner product.

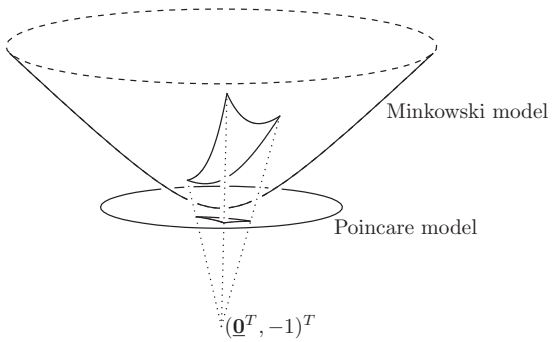

Figure 2.

Lemma 5. *For distinct* $\mathbf{a}$, $\mathbf{b} \in \mathbb{H}^{n-1}$, *the linear transformation* $\varphi(\,\cdot\,;\mathbf{a},\mathbf{b})$ *of* $\mathbb{R}^n$ *defined by*

$$\varphi(\mathbf{x};\mathbf{a},\mathbf{b}) = \Big(\mathbf{x} + \langle\mathbf{x}|\mathbf{a}\rangle\mathbf{a} - \frac{\big\langle\mathbf{x}\big|\mathbf{b} + \langle\mathbf{a}|\mathbf{b}\rangle\mathbf{a}\big\rangle}{\langle\mathbf{a}|\mathbf{b}\rangle^2 - 1}\big(\mathbf{b} + \langle\mathbf{a}|\mathbf{b}\rangle\mathbf{a}\big)\Big) - \langle\mathbf{x}|\mathbf{a}\rangle\mathbf{b} + \frac{\big\langle\mathbf{x}\big|\mathbf{b} + \langle\mathbf{a}|\mathbf{b}\rangle\mathbf{a}\big\rangle}{\langle\mathbf{a}|\mathbf{b}\rangle^2 - 1}\big(-\langle\mathbf{a}|\mathbf{b}\rangle\mathbf{b} - \mathbf{a}\big) \tag{22}$$

maps $\mathbf{a}$ *to* $\mathbf{b}$, *preserves the pseudo-inner product* $\langle\cdot|\cdot\rangle$ *of* $\mathbb{R}^n$, *and maps* $\mathbb{H}^{n-1}$ *onto itself. Remark that* $-\langle\mathbf{a}\,|\,\mathbf{b}\rangle > 1$ *from Lemma* 3 (*precisely, we have* $-\langle\mathbf{a}\,|\,\mathbf{b}\rangle = \cosh d_{\mathbb{H}^{n-1}}(\mathbf{a},\mathbf{b})$).

Proof. Let $\mathbf{v}$ and $\mathbf{w}$ be the vectors of $\mathbb{R}\cdot\mathbf{a} + \mathbb{R}\cdot\mathbf{b}$ such that

$$\langle\mathbf{a}|\mathbf{v}\rangle = 0 = \langle\mathbf{b}|\mathbf{w}\rangle, \quad \langle\mathbf{b}|\mathbf{v}\rangle > 0 > \langle\mathbf{a}|\mathbf{w}\rangle, \quad \text{and} \quad \langle\mathbf{v}|\mathbf{v}\rangle = 1 = \langle\mathbf{w}|\mathbf{w}\rangle, \tag{23}$$

i.e.,

$$\mathbf{v} = \frac{\mathbf{b} + \langle\mathbf{a}|\mathbf{b}\rangle\mathbf{a}}{\sqrt{\langle\mathbf{a}|\mathbf{b}\rangle^2 - 1}} \quad \text{and} \quad \mathbf{w} = \frac{-\langle\mathbf{a}|\mathbf{b}\rangle\mathbf{b} - \mathbf{a}}{\sqrt{\langle\mathbf{a}|\mathbf{b}\rangle^2 - 1}}. \tag{24}$$

(In other words, $\mathbf{v}$ (resp. $-\mathbf{w}$) is the tangent vector at $\mathbf{a}$ (resp. $\mathbf{b}$) of the geodesic between $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{H}^{n-1}$ with $\langle\mathbf{v}|\mathbf{v}\rangle = 1$ (resp. $\langle\mathbf{w}|\mathbf{w}\rangle = 1$).) Then the transformation φ is represented by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$, and $\mathbf{w}$:

$$\varphi(\mathbf{x};\mathbf{a},\mathbf{b}) = \Big(\mathbf{x} + \langle\mathbf{x}|\mathbf{a}\rangle\mathbf{a} - \langle\mathbf{x}|\mathbf{v}\rangle\mathbf{v}\Big) - \langle\mathbf{x}|\mathbf{a}\rangle\mathbf{b} + \langle\mathbf{x}|\mathbf{v}\rangle\mathbf{w}. \tag{25}$$

We have $\varphi(\mathbf{a};\mathbf{a},\mathbf{b}) = \Big(\mathbf{a} + (-\mathbf{a}) - \mathbf{0}\Big) - (-\mathbf{b}) + \mathbf{0} = \mathbf{b}$ and

$$\begin{aligned}
&\langle\varphi(\mathbf{x};\mathbf{a},\mathbf{b})|\varphi(\mathbf{y};\mathbf{a},\mathbf{b})\rangle \\
&= \Big\langle\mathbf{x} + \big(\langle\mathbf{x}|\mathbf{a}\rangle\mathbf{a} - \langle\mathbf{x}|\mathbf{v}\rangle\mathbf{v}\big)\Big|\mathbf{y} + \big(\langle\mathbf{y}|\mathbf{a}\rangle\mathbf{a} - \langle\mathbf{y}|\mathbf{v}\rangle\mathbf{v}\big)\Big\rangle \\
&\quad + \langle\mathbf{x}|\mathbf{a}\rangle\langle\mathbf{y}|\mathbf{a}\rangle\langle\mathbf{b}|\mathbf{b}\rangle + \langle\mathbf{x}|\mathbf{v}\rangle\langle\mathbf{y}|\mathbf{v}\rangle\langle\mathbf{w}|\mathbf{w}\rangle \\
&= \Big(\langle\mathbf{x}|\mathbf{y}\rangle + 2\big(\langle\mathbf{x}|\mathbf{a}\rangle\langle\mathbf{y}|\mathbf{a}\rangle - \langle\mathbf{x}|\mathbf{v}\rangle\langle\mathbf{y}|\mathbf{v}\rangle\big) + \big(\langle\mathbf{x}|\mathbf{a}\rangle\langle\mathbf{y}|\mathbf{a}\rangle\cdot(-1) + \langle\mathbf{x}|\mathbf{v}\rangle\langle\mathbf{y}|\mathbf{v}\rangle\cdot 1\big)\Big) \\
&\quad + \langle\mathbf{x}|\mathbf{a}\rangle\langle\mathbf{y}|\mathbf{a}\rangle\cdot(-1) + \langle\mathbf{x}|\mathbf{v}\rangle\langle\mathbf{y}|\mathbf{v}\rangle\cdot 1 \\
&= \langle\mathbf{x}|\mathbf{y}\rangle,
\end{aligned}$$

where the first equality comes from the pseudo-orthogonality of three terms of the right-hand side of (25) and the second equality comes from $\langle \mathbf{a}\,|\,\mathbf{a}\rangle = -1 = \langle \mathbf{b}|\mathbf{b}\rangle$ and (23). So $\varphi(\mathbb{H}^{n-1}\cup\iota(\mathbb{H}^{n-1});\mathbf{a},\mathbf{b}) = \mathbb{H}^{n-1}\cup\iota(\mathbb{H}^{n-1})$. The restriction $\varphi(\cdot\,;\mathbf{a},\mathbf{b})|_{\mathbb{H}^{n-1}\cup\iota(\mathbb{H}^{n-1})}$ is homotopic to $\mathrm{id}_{\mathbb{H}^{n-1}\cup\iota(\mathbb{H}^{n-1})}$ by $\{\varphi(\cdot\,;\mathbf{a},(1-t)\mathbf{a}\oplus_{\mathbb{H}^{n-1}} t\mathbf{b})|_{\mathbb{H}^{n-1}\cup\iota(\mathbb{H}^{n-1})} : t\in[0,1]\}$ where $(1-t)\mathbf{a}\oplus_{\mathbb{H}^{n-1}} t\mathbf{b}$ is the internally dividing point of the geodesic between $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{H}^{n-1}$ in the ratio of $t:(1-t)$ (we can represent it by

$$\begin{aligned}&(1-t)\mathbf{a}\oplus_{\mathbb{H}^{n-1}} t\mathbf{b}\\ =\;&\cosh\big(td_{\mathbb{H}^{n-1}}(\mathbf{a},\mathbf{b})\big)\mathbf{a}+\sinh\big(td_{\mathbb{H}^{n-1}}(\mathbf{a},\mathbf{b})\big)\mathbf{v}\\ =\;&\cosh\big((1-t)d_{\mathbb{H}^{n-1}}(\mathbf{a},\mathbf{b})\big)\mathbf{b}-\sinh\big((1-t)d_{\mathbb{H}^{n-1}}(\mathbf{a},\mathbf{b})\big)\mathbf{w},\end{aligned}\tag{26}$$

precisely). $\mathbb{H}^{n-1}$ and $\iota(\mathbb{H}^{n-1})$ are disconnected, so $\varphi(\mathbb{H}^{n-1};\mathbf{a},\mathbf{b}) = \mathbb{H}^{n-1}$. □

Let S be a hyperbolic simplex of $\mathbb{H}^{n-1}$ with vertices $\mathbf{p}_0^*,\ldots,\mathbf{p}_{n-1}^*\in\mathbb{H}^{n-1}$, i.e.,

$$S=\Big(\mathbb{R}^+\cdot\mathbf{p}_0^*+\cdots+\mathbb{R}^+\cdot\mathbf{p}_{n-1}^*\Big)\cap\mathbb{H}^{n-1},\tag{27}$$

where $\mathbb{R}^+$ is the set of non-negative real numbers. For an index $j=0,\ldots,n-1$, if $\mathbf{b}\in\mathbb{R}^n\setminus\{\mathbf{0}\}$ satisfies $\langle\mathbf{p}_i^*|\mathbf{b}\rangle = 0$ for $i=0,\ldots,\widehat{j},\ldots,n-1$, then $\langle\mathbf{p}_j^*|\mathbf{b}\rangle\neq 0$ and $\langle\mathbf{b}|\mathbf{b}\rangle>0$ from Corollary 4. Let $\mathbf{p}_j\in\mathbb{R}^n\setminus\{\mathbf{0}\}$ be one of the normalizations of such $\mathbf{b}$, i.e., let $\mathbf{p}_j$ be the unique vector such that

$$\langle\mathbf{p}_i^*|\mathbf{p}_j\rangle=0\quad\text{for}\quad i=0,\ldots,\widehat{j},\ldots,n-1;\quad\langle\mathbf{p}_j^*|\mathbf{p}_j\rangle>0;\quad\text{and}\quad\langle\mathbf{p}_j|\mathbf{p}_j\rangle=1.\tag{28}$$

Then,

$$S=\{\mathbf{x}\in\mathbb{H}^{n-1}:\langle\mathbf{x}|\mathbf{p}_i\rangle\geqq 0\quad\forall i=0,\ldots,n-1\},\tag{29}$$

because of the following equivalence for $\mathbf{x}=\sum_{i=0}^{n-1}\alpha_i\mathbf{p}_i^*\in\mathbb{H}^{n-1}$:

$$\mathbf{x}\in S\Longleftrightarrow\alpha_i\geqq 0\quad\forall i\Longleftrightarrow\alpha_i\langle\mathbf{p}_i^*|\mathbf{p}_i\rangle\geqq 0\quad\forall i\Longleftrightarrow\langle\mathbf{x}|\mathbf{p}_i\rangle\geqq 0\quad\forall i.$$

Remark 2. From Lemma 5, we can assume that one vertex of S is $(\underline{\mathbf{0}}^T,1)^T$, without loss of generality (the fact is similar to the spherical simplex, in particular, we can assume that one vertex of the spherical simplex S' of $\mathbb{S}^{n-1}=\{\mathbf{x}'\in\mathbb{R}^n : \mathbf{x}'\,.\,\mathbf{x}'=1\}$ is $(\underline{\mathbf{0}}^T,1)^T$, without loss of generality).

The symbols Δ^* and Δ mean

$$\Delta^*\begin{pmatrix}k_0\cdots k_\ell & i_{\ell+1}\cdots i_m\\ & j_{\ell+1}\cdots j_m\end{pmatrix}=\det(-\langle\mathbf{p}_r^*|\mathbf{p}_s^*\rangle)_{s=k_0,\ldots,k_\ell,j_{\ell+1},\ldots,j_m}^{r=k_0,\ldots,k_\ell,i_{\ell+1},\ldots,i_m},\tag{30}^*$$

$$\Delta\begin{pmatrix}k_0\cdots k_\ell & i_{\ell+1}\cdots i_m\\ & j_{\ell+1}\cdots j_m\end{pmatrix}=\det(\langle\mathbf{p}_r|\mathbf{p}_s\rangle)_{s=k_0,\ldots,k_\ell,j_{\ell+1},\ldots,j_m}^{r=k_0,\ldots,k_\ell,i_{\ell+1},\ldots,i_m},\tag{30}$$

for integers $\ell, m=-1,\ldots,n-1$ with $\ell\leqq m$ and indices $k_0,\ldots,k_\ell$, $i_{\ell+1},\ldots,i_m$, $j_{\ell+1},\ldots,j_m=0,\ldots,n-1$ (notice that left upper diagonal elements of matrices of the right-hand sides of the above, $-\langle\mathbf{p}_k^*\,|\,\mathbf{p}_k^*\rangle$ and $\langle\mathbf{p}_k\,|\,\mathbf{p}_k\rangle$ for $k=k_0,\ldots,k_\ell$, are equal to 1). For symbols above, we have following three lemmas, which are similar to lemmas of [6] and [7].

Lemma 6. *For an integer $m = -1,\ldots, n-1$ and pairwise distinct indices $k_0, \ldots, k_m = 0, \ldots, n-1$, we have the following inequalities:*

$$(-1)^m \Delta^*(k_0 \cdots k_m) \begin{cases} > 0 & (0 \leqq m \leqq n-1), \\ < 0 & (m = -1), \end{cases} \tag{31}^*$$

$$\Delta(k_0 \cdots k_m) \begin{cases} > 0 & (-1 \leqq m \leqq n-2), \\ < 0 & (m = n-1). \end{cases} \tag{31}$$

Proof. It is enough to show the inequalities above for $k_r = r$ for $r = 0,\ldots, m$. For $m \geqq 0$, we can assume $\mathbf{p}^*_m = (\underline{\mathbf{0}}^T, 1)^T$, so we have

$$(-1)^m \Delta^*(0 \cdots m) = (-1)^m \det(-\langle \mathbf{p}^*_r | \mathbf{p}^*_s \rangle)^{r=0,\ldots,m}_{s=0,\ldots,m} = \det(\mathbf{p}^*_r \cdot \mathbf{p}^*_s)^{r=0,\ldots,m}_{s=0,\ldots,m} > 0.$$

where the second equality comes from (16). For $m = -1$, it is trivial:

$$(-1)^{-1} \Delta^*() = -1 < 0.$$

For $m \leqq n-2$, we can assume $\mathbf{p}^*_{n-1} = (\underline{\mathbf{0}}^T, 1)^T$, so $\mathbf{p}_0,\ldots, \mathbf{p}_m$ are in $\mathbb{R}^{n-1} \times \{0\}$. Hence, it is also trivial:

$$\Delta(0 \cdots m) = \det(\langle \mathbf{p}_r | \mathbf{p}_s \rangle)^{r=0,\ldots,m}_{s=0,\ldots,m} = \det(\mathbf{p}_r \cdot \mathbf{p}_s)^{r=0,\ldots,m}_{s=0,\ldots,m} > 0.$$

For $m = n-1$, we have

$$\Delta(0 \cdots n-1) = \det(\langle \mathbf{p}_r | \mathbf{p}_s \rangle)^{r=0,\ldots,n-1}_{s=0,\ldots,n-1} = -(\det(\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}))^2 < 0,$$

where the second equality comes from (14). □

Lemma 7. *Let*

$$\varepsilon^* = \operatorname{sgn} \det(\mathbf{p}^*_0, \ldots, \mathbf{p}^*_{n-1}). \tag{32}^*$$

Then we have

$$-\varepsilon^* = \operatorname{sgn} \det(\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}), \tag{32}$$

$$\mathbf{p}_k = (-1)^{n-1-k} \varepsilon^* \frac{\iota(\langle\!\langle \mathbf{p}^*_0, \ldots, \widehat{\mathbf{p}^*_k}, \ldots, \mathbf{p}^*_{n-1} \rangle\!\rangle)}{\sqrt{(-1)^{n-2} \Delta^*(0 \cdots \widehat{k} \cdots n-1)}}, \tag{33}^*$$

$$\mathbf{p}^*_k = (-1)^{n-1-k} (-\varepsilon^*) \frac{\iota(\langle\!\langle \mathbf{p}_0, \ldots, \widehat{\mathbf{p}_k}, \ldots, \mathbf{p}_{n-1} \rangle\!\rangle)}{\sqrt{\Delta(0 \cdots \widehat{k} \cdots n-1)}}, \tag{33}$$

$$\langle \mathbf{p}^*_k | \mathbf{p}_k \rangle = \frac{\sqrt{(-1)^{n-1} \Delta^*(0 \cdots n-1)}}{\sqrt{(-1)^{n-2} \Delta^*(0 \cdots \widehat{k} \cdots n-1)}}, \tag{34}^*$$

$$\langle \mathbf{p}^*_k | \mathbf{p}_k \rangle = \frac{\sqrt{-\Delta(0 \cdots n-1)}}{\sqrt{\Delta(0 \cdots \widehat{k} \cdots n-1)}}, \tag{34}$$

for each index $k = 0,\ldots, n-1$.

Proof. From $(32)^*$, $\mathbf{p}_k$ and $(-1)^{n-1-k}\varepsilon^*\iota(\langle\langle\mathbf{p}_0^*,\dots,\widehat{\mathbf{p}_k^*},\dots,\mathbf{p}_{n-1}^*\rangle\rangle)$ are parallel and have the same direction, so, we have $(33)^*$ from $\langle\mathbf{p}_k|\mathbf{p}_k\rangle = 1$ and

$$\begin{aligned}
&\langle\iota(\langle\langle\mathbf{p}_0^*,\dots,\widehat{\mathbf{p}_k^*},\dots,\mathbf{p}_{n-1}^*\rangle\rangle)|\iota(\langle\langle\mathbf{p}_0^*,\dots,\widehat{\mathbf{p}_k^*},\dots,\mathbf{p}_{n-1}^*\rangle\rangle)\rangle\\
&=\langle\langle\langle\mathbf{p}_0^*,\dots,\widehat{\mathbf{p}_k^*},\dots,\mathbf{p}_{n-1}^*\rangle\rangle|\langle\langle\mathbf{p}_0^*,\dots,\widehat{\mathbf{p}_k^*},\dots,\mathbf{p}_{n-1}^*\rangle\rangle\rangle\\
&=(-1)^{n-2}\det(-\langle\mathbf{p}_r^*|\mathbf{p}_s^*\rangle)_{s=0,\dots,\widehat{k},\dots,n-1}^{r=0,\dots,\widehat{k},\dots,n-1}\\
&=(-1)^{n-2}\Delta^*(0,\dots,\widehat{k},\dots,n-1),
\end{aligned}\tag{35}$$

where the second equality comes from (15). We also have (32):

$$\begin{aligned}
&\operatorname{sgn}\det(\mathbf{p}_0,\dots,\mathbf{p}_\ell,\dots,\mathbf{p}_{n-1})\\
&=\operatorname{sgn}\det(\dots,(-1)^{n-1-\ell}\varepsilon^*\iota(\langle\langle\mathbf{p}_0^*,\dots,\widehat{\mathbf{p}_\ell^*},\dots,\mathbf{p}_{n-1}^*\rangle\rangle),\dots)\\
&=(\varepsilon^*)^n\operatorname{sgn}\det(\dots,(-1)^{n-1-\ell}\iota(\langle\langle\mathbf{p}_0^*,\dots,\widehat{\mathbf{p}_\ell^*},\dots,\mathbf{p}_{n-1}^*\rangle\rangle),\dots)\\
&=-(\varepsilon^*)^n\operatorname{sgn}\det(\dots,(-1)^{n-1-\ell}\langle\langle\mathbf{p}_0^*,\dots,\widehat{\mathbf{p}_\ell^*},\dots,\mathbf{p}_{n-1}^*\rangle\rangle,\dots)\\
&=-(\varepsilon^*)^n\operatorname{sgn}(\det(\mathbf{p}_0^*,\dots,\mathbf{p}_{n-1}^*))^{n-1}\\
&=-\varepsilon^*,
\end{aligned}$$

where the first and fourth equalities come from $(33)^*$ and (4), respectively. From (32), $\mathbf{p}_k^*$ and $(-1)^{n-1-k}(-\varepsilon^*)\iota(\langle\langle\mathbf{p}_0,\dots\widehat{\mathbf{p}_k}\dots,\mathbf{p}_{n-1}\rangle\rangle)$ are parallel and have the same direction, so, we have (33) from $\langle\mathbf{p}_k^*|\mathbf{p}_k^*\rangle = -1$ and

$$\begin{aligned}
&-\langle\iota(\langle\langle\mathbf{p}_0,\dots,\widehat{\mathbf{p}_k},\dots,\mathbf{p}_{n-1}\rangle\rangle)|\iota(\langle\langle\mathbf{p}_0,\dots,\widehat{\mathbf{p}_k},\dots,\mathbf{p}_{n-1}\rangle\rangle)\rangle\\
&=-\langle\langle\langle\mathbf{p}_0,\dots,\widehat{\mathbf{p}_k},\dots,\mathbf{p}_{n-1}\rangle\rangle|\langle\langle\mathbf{p}_0,\dots,\widehat{\mathbf{p}_k},\dots,\mathbf{p}_{n-1}\rangle\rangle\rangle\\
&=\det(\langle\mathbf{p}_r|\mathbf{p}_s\rangle)_{s=0,\dots,\widehat{k},\dots,n-1}^{r=0,\dots,\widehat{k},\dots,n-1}\\
&=\Delta(0,\dots,\widehat{k},\dots,n-1),
\end{aligned}\tag{36}$$

where the second equality comes from (15). We have $(34)^*$ from $(33)^*$ and

$$\begin{aligned}
&\langle\mathbf{p}_k^*|(-1)^{n-1-k}\varepsilon^*\iota(\langle\langle\mathbf{p}_0^*,\dots,\widehat{\mathbf{p}_k^*},\dots,\mathbf{p}_{n-1}^*\rangle\rangle)\rangle\\
&=|\det(\mathbf{p}_0^*,\dots,\mathbf{p}_{n-1}^*)|\\
&=\sqrt{(-1)^{n-1}\det(-\langle\mathbf{p}_r^*|\mathbf{p}_s^*\rangle)_{s=0,\dots,n-1}^{r=0,\dots,n-1}}\\
&=\sqrt{(-1)^{n-1}\Delta^*(0,\dots,n-1)},
\end{aligned}\tag{37}$$

where the first and second equalities come from $(32)^*$ and (14), respectively. We also have (34) from (33) and

$$\begin{aligned}
&\langle(-1)^{n-1-k}(-\varepsilon^*)\iota(\langle\langle\mathbf{p}_0,\dots,\widehat{\mathbf{p}_k},\dots,\mathbf{p}_{n-1}\rangle\rangle)|\mathbf{p}_k\rangle\\
&=|\det(\mathbf{p}_0,\dots,\mathbf{p}_{n-1})|\\
&=\sqrt{-\det(\langle\mathbf{p}_r|\mathbf{p}_s\rangle)_{s=0,\dots,n-1}^{r=0,\dots,n-1}}\\
&=\sqrt{-\Delta(0,\dots,n-1)},
\end{aligned}\tag{38}$$

where the first and second equalities come from (32) and (14), respectively. □

Lemma 8. *Let $\{k_0, \dots, k_{n-1}\}$ be a permutation of the set of all indices $\{0, \dots, n-1\}$. Then we have*

$$\Delta(k_0 \cdots k_m) = \frac{(-1)^{n-2-m}\Delta^*(k_{m+1} \cdots k_{n-1})}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \prod_{r=0}^{m} \langle \mathbf{p}^*_{k_r} | \mathbf{p}_{k_r} \rangle^2, \tag{39}^*$$

$$(-1)^m \Delta^*(k_0 \cdots k_m) = \frac{\Delta(k_{m+1} \cdots k_{n-1})}{-\Delta(0 \cdots n-1)} \prod_{r=0}^{m} \langle \mathbf{p}^*_{k_r} | \mathbf{p}_{k_r} \rangle^2, \tag{39}$$

for an integer $m = -1, \dots, n-1$, and we also have

$$\begin{aligned} &-\Delta \begin{pmatrix} k_0 \cdots k_{m-1} & i \\ & j \end{pmatrix} \\ &= \frac{(-1)^{n-2-m}\Delta^* \begin{pmatrix} k_m \cdots \widehat{i} \cdots \widehat{j} \cdots k_{n-1} & j \\ & i \end{pmatrix}}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \cdot \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \prod_{r=0}^{m-1} \langle \mathbf{p}^*_{k_r} | \mathbf{p}_{k_r} \rangle^2, \end{aligned} \tag{40}^*$$

$$\begin{aligned} &(-1)^m \Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} & i \\ & j \end{pmatrix} \\ &= \frac{-\Delta \begin{pmatrix} k_m \cdots \widehat{i} \cdots \widehat{j} \cdots k_{n-1} & j \\ & i \end{pmatrix}}{-\Delta(0 \cdots n-1)} \cdot \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \prod_{r=0}^{m-1} \langle \mathbf{p}^*_{k_r} | \mathbf{p}_{k_r} \rangle^2, \end{aligned} \tag{40}$$

for an integer $m = 0, \dots, n-2$ and distinct indices $i, j = k_m, \dots, k_{n-1}$ (we use the notation $k_m \cdots \widehat{k_s} \cdots \widehat{k_t} \cdots k_{n-1}$ whether $s < t$ or not. If $s > t$, it means $k_m \cdots \widehat{k_t} \cdots \widehat{k_s} \cdots k_{n-1}$).

Proof. (39) and (40) are natural consequences of (39)* and (40)*, so we only have to prove (39)* and (40)*. To prove them, it is enough to show the following four equations:

$$\Delta() = \frac{(-1)^{n-1}\Delta^*(0 \cdots n-1)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \cdot 1, \tag{41}$$

$$\Delta(0 \cdots n-1) = \frac{(-1)^{-1}\Delta^*()}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \prod_{r=0}^{n-1} \langle \mathbf{p}^*_r | \mathbf{p}_r \rangle^2, \tag{42}$$

$$\begin{aligned} -\Delta \begin{pmatrix} 0 \cdots n-3 & n-2 \\ & n-1 \end{pmatrix} &= \frac{(-1)^0 \Delta^* \binom{n-1}{n-2}}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \\ &\quad \cdot \langle \mathbf{p}^*_{n-2} | \mathbf{p}_{n-2} \rangle \langle \mathbf{p}^*_{n-1} | \mathbf{p}_{n-1} \rangle \prod_{r=0}^{n-3} \langle \mathbf{p}^*_r | \mathbf{p}_r \rangle^2, \end{aligned} \tag{43}$$

$$\begin{aligned} \Delta \begin{pmatrix} 0 \cdots m-1 & i \\ & j \end{pmatrix} &= \frac{(-1)^{n-2-m}(-1)^{j-i}\Delta^* \binom{m \cdots \widehat{i} \cdots n-1}{m \cdots \widehat{j} \cdots n-1}}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \cdot \\ &\quad \cdot \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \prod_{r=0}^{m-1} \langle \mathbf{p}^*_r | \mathbf{p}_r \rangle^2, \end{aligned} \tag{44}$$

for m, i, $j = 0, \dots, n-1$ with $\{i, j\} \subsetneqq \{m, \dots, n-1\}$. In particular, (41) implies (39)* for $m = -1$, (42) implies (39)* for $m = n-1$, (43) implies (40)*

for $m = n-2$, and (44) implies (39)* for $0 \leqq m \leqq n-2$ and the sign inversion of (40)* for $0 \leqq m \leqq n-3$ because

$$(-1)^{j-i}\Delta^*\begin{pmatrix} m\cdots\widehat{i}\cdots n-1 \\ m\cdots\widehat{j}\cdots n-1\end{pmatrix} = \begin{cases} \Delta^*(m\cdots\widehat{i}\cdots n-1) & \text{if } i=j, \\ -\Delta^*\begin{pmatrix} m\cdots\widehat{i}\cdots\widehat{j}\cdots n-1\, j \\ i\end{pmatrix} & \text{if } i\neq j. \end{cases} \tag{45}$$

The equation (41) is obvious.
We will show (42) and (43) by (44) and afterwards we will show (44). We have (42) immediately:

$$\begin{aligned}\Delta(0\cdots n-1) &= -\langle \mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle^2\cdot\Delta(0\cdots n-2)\\ &= -\langle \mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle^2\cdot\frac{(-1)^0\Delta^*(n-1)}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\prod_{r=0}^{n-2}\langle \mathbf{p}^*_r|\mathbf{p}_r\rangle^2\\ &= \frac{(-1)^{-1}\Delta^*()}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\prod_{r=0}^{n-1}\langle \mathbf{p}^*_r|\mathbf{p}_r\rangle^2,\end{aligned}$$

where the first equality comes from (34) and the second equality comes from (44) for $m=i=j=n-2$. We also have (43) by

$$\begin{aligned}&-\Delta\begin{pmatrix} 0\cdots n-3\, n-2\\ n-1\end{pmatrix}\\ &= \langle\langle\!\langle \mathbf{p}_0,\ldots,\mathbf{p}_{n-2}\rangle\!\rangle\!\rangle|\langle\langle\!\langle \mathbf{p}_0,\ldots,\mathbf{p}_{n-3},\mathbf{p}_{n-1}\rangle\!\rangle\!\rangle\rangle\\ &= -\langle(-1)^0(-\varepsilon^*)\iota(\langle\langle\!\langle \mathbf{p}_0,\ldots,\mathbf{p}_{n-2}\rangle\!\rangle\!\rangle)|(-1)^1(-\varepsilon^*)\iota(\langle\langle\!\langle \mathbf{p}_0,\ldots,\mathbf{p}_{n-3},\mathbf{p}_{n-1}\rangle\!\rangle\!\rangle)\rangle\\ &= -\langle\sqrt{\Delta(0\cdots n-2)}\mathbf{p}^*_{n-1}|\sqrt{\Delta(0\cdots n-3,n-1)}\mathbf{p}^*_{n-2}\rangle\\ &= \sqrt{\Delta(0\cdots n-2)}\cdot\sqrt{\Delta(0\cdots n-3,n-1)}\cdot(-\langle \mathbf{p}^*_{n-1}|\mathbf{p}^*_{n-2}\rangle)\\ &= \frac{\sqrt{(-1)^0\Delta^*(n-1)}}{\sqrt{(-1)^{n-1}\Delta^*(0\cdots n-1)}}\prod_{r=0}^{n-2}\langle \mathbf{p}^*_r|\mathbf{p}_r\rangle\\ &\quad\cdot\frac{\sqrt{(-1)^0\Delta^*(n-2)}}{\sqrt{(-1)^{n-1}\Delta^*(0\cdots n-1)}}\prod_{r=0}^{\substack{r\neq n-2\\ n-1}}\langle \mathbf{p}^*_r|\mathbf{p}_r\rangle\cdot\Delta^*\begin{pmatrix} n-1\\ n-2\end{pmatrix}\\ &= \frac{(-1)^0\Delta^*\begin{pmatrix} n-1\\ n-2\end{pmatrix}}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\langle \mathbf{p}^*_{n-2}|\mathbf{p}_{n-2}\rangle\langle \mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle\prod_{r=0}^{n-3}\langle \mathbf{p}^*_r|\mathbf{p}_r\rangle^2,\end{aligned}$$

where the first equality comes from (15), the third equality comes from (33), and the fifth equality comes from (44) for $m=i=j=n-2$ and for $m=n-2$ and $i=j=n-1$. Now it remains only to show (44). To show it, we can assume $i\neq n-1\neq j$ and $\mathbf{p}^*_{n-1}=(\underline{\mathbf{0}}^T,1)^T$. Let

$$\mathbf{p}'^*_k = \frac{\mathbf{p}^*_k}{\sqrt{\mathbf{p}^*_k\cdot\mathbf{p}^*_k}} \tag{46}$$

for $k=0,\ldots,n-1$ and $\mathbf{p}'_\ell\in\mathbb{R}^n$ be such that

$$\mathbf{p}'^*_k\cdot\mathbf{p}'_\ell = 0 \quad\text{for}\quad k=0,\ldots\widehat{\ell}\ldots,n-1; \quad \mathbf{p}'^*_\ell\cdot\mathbf{p}'_\ell>0; \quad\text{and}\quad \mathbf{p}'_\ell\cdot\mathbf{p}'_\ell=1; \tag{47}$$

for $\ell = 0,\ldots, n-1$ (notice that

$$\mathbf{p}'^*_\ell \cdot \mathbf{p}'_\ell = \frac{\langle \mathbf{p}^*_\ell | \mathbf{p}_\ell \rangle}{\sqrt{\mathbf{p}^*_\ell \cdot \mathbf{p}^*_\ell}}, \tag{48}$$

$$\mathbf{p}'_k \cdot \mathbf{p}'_\ell = \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle, \tag{49}$$

for $k,\ \ell = 0,\ldots,\ n-2$, because $\mathbf{p}'^*_{n-1} = \mathbf{p}^*_{n-1} = (\underline{\mathbf{0}}^T, 1)^T$ and $\mathbf{p}'_\ell = \mathbf{p}_\ell \in \mathbb{R}^{n-1} \times \{0\}$). Then

$$S' = \left(\mathbb{R}^+ \cdot \mathbf{p}'^*_0 + \cdots + \mathbb{R}^+ \cdot \mathbf{p}'^*_{n-1}\right) \cap \mathbb{S}^{n-1} = \{\mathbf{x}' \in \mathbb{S}^{n-1} : \mathbf{x}' . \mathbf{p}'_\ell \geqq 0, \forall \ell = 0, \ldots, n-1\}$$

is a spherical simplex on the unit sphere $\mathbb{S}^{n-1}$ and we denote

$$\Delta'^*_{\mathbb{S}^{n-1}}\begin{pmatrix} k_0 \cdots k_\ell & i_{\ell+1} \cdots i_h \\ & j_{\ell+1} \cdots j_h \end{pmatrix} = \det(\mathbf{p}'^*_r \cdot \mathbf{p}'^*_s)^{r=k_0,\ldots,k_\ell,i_{\ell+1},\ldots,i_h}_{s=k_0,\ldots,k_\ell,j_{\ell+1},\ldots,j_h},$$

$$\Delta'_{\mathbb{S}^{n-1}}\begin{pmatrix} k_0 \cdots k_\ell & i_{\ell+1} \cdots i_h \\ & j_{\ell+1} \cdots j_h \end{pmatrix} = \det(\mathbf{p}'_r \cdot \mathbf{p}'_s)^{r=k_0,\ldots,k_\ell,i_{\ell+1},\ldots,i_h}_{s=k_0,\ldots,k_\ell,j_{\ell+1},\ldots,j_h}.$$

We have

$$\begin{aligned} &\Delta'^*_{\mathbb{S}^{n-1}}\begin{pmatrix} m \cdots \widehat{i} \cdots n-1 \\ m \cdots \widehat{j} \cdots n-1 \end{pmatrix} \\ &= \frac{\sqrt{\mathbf{p}^*_i \cdot \mathbf{p}^*_i}\sqrt{\mathbf{p}^*_j \cdot \mathbf{p}^*_j}}{\prod_{r=m}^{n-1} (\mathbf{p}^*_r \cdot \mathbf{p}^*_r)} \det(\mathbf{p}^*_k \cdot \mathbf{p}^*_\ell)^{k=m\cdots\widehat{i}\cdots n-1}_{\ell=m\cdots\widehat{j}\cdots n-1} \\ &= \frac{(-1)^{n-2-m}\sqrt{\mathbf{p}^*_i \cdot \mathbf{p}^*_i}\sqrt{\mathbf{p}^*_j \cdot \mathbf{p}^*_j}}{\prod_{r=m}^{n-1} (\mathbf{p}^*_r \cdot \mathbf{p}^*_r)} \Delta^* \begin{pmatrix} m \cdots \widehat{i} \cdots n-1 \\ m \cdots \widehat{j} \cdots n-1 \end{pmatrix} \end{aligned} \tag{50}$$

where the former and latter equalities come from (46) and (16), respectively, and we also have similarly

$$\Delta'^*_{\mathbb{S}^{n-1}}(0 \cdots n-1) = \frac{(-1)^{n-1}}{\prod_{r=0}^{n-1} (\mathbf{p}^*_r \cdot \mathbf{p}^*_r)} \Delta^*(0 \cdots n-1). \tag{51}$$

So, we have (44):

$$\begin{aligned} &\Delta\begin{pmatrix} 0 \cdots m-1 & i \\ & j \end{pmatrix} = \Delta'_{\mathbb{S}^{n-1}}\begin{pmatrix} 0 \cdots m-1 & i \\ & j \end{pmatrix} \\ &= \frac{(-1)^{j-i}\Delta'^*_{\mathbb{S}^{n-1}}\begin{pmatrix} m \cdots \widehat{i} \cdots n-1 \\ m \cdots \widehat{j} \cdots n-1 \end{pmatrix}}{\Delta'^*_{\mathbb{S}^{n-1}}(0 \cdots n-1)} \cdot (\mathbf{p}'^*_i \cdot \mathbf{p}'_i)(\mathbf{p}'^*_j \cdot \mathbf{p}'_j) \prod_{r=0}^{m-1} (\mathbf{p}'^*_r \cdot \mathbf{p}'_r)^2 \\ &= \frac{(-1)^{n-2-m}(-1)^{j-i}\Delta^* \begin{pmatrix} m \cdots \widehat{i} \cdots n-1 \\ m \cdots \widehat{j} \cdots n-1 \end{pmatrix}}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \prod_{r=0}^{m-1} \langle \mathbf{p}^*_r | \mathbf{p}_r \rangle^2, \end{aligned}$$

where the first equality comes from (49), the second equality comes from Lemma 5.3 of [6] and (45) or from Remark 8 of [7] and (45), and the last equality comes from (48), (50), and (51). □

4. Orthocentric simplices

Hereafter, we consider orthocenters of simplices.

Definition 1. *The point* $\mathbf{n} \in \mathbb{H}^{n-1}$ *is called the orthocenter of* S *if* $\mathbf{n}$ *is a concurrent point of geodesics* C_i *for all* $i = 0,\ldots, n-1$*, where* C_i *is the pseudo-perpendicular to the opposite face*

$$\{\mathbf{x} \in S : \langle \mathbf{x}|\mathbf{p}_i\rangle = 0\} = \left(\mathbb{R}^+ \cdot \mathbf{p}_0^* + \cdots + \widehat{\mathbb{R}^+ \cdot \mathbf{p}_i^*} + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}^*\right) \cap \mathbb{H}^{n-1} \quad (52)$$

of the vertex $\mathbf{p}_i^*$*, in other words,* $\mathbf{n} \in \mathbb{R} \cdot \mathbf{p}_i^* + \mathbb{R} \cdot \mathbf{p}_i$ *for all* i*. A simplex having orthocenter is called an orthocentric simplex.*

Remark 3. If the orthocenter exists, it is unique (unlike in the case of the spherical simplex, it is impossible that $\mathbf{p}_i^* = \mathbf{p}_i$ for any index $i = 0,\ldots, n-1$, so, the concurrent point of $(\mathbb{R} \cdot \mathbf{p}_i^* + \mathbb{R} \cdot \mathbf{p}_i) \cap \mathbb{H}^{n-1}$ is unique, if it exists).

The conditions below hold an important role for orthocentric simplices.

Lemma 9. *The followings are equivalent:*

(a)* $\Delta^* \begin{pmatrix} i \\ k \end{pmatrix} \Delta^* \begin{pmatrix} j \\ \ell \end{pmatrix} = \Delta^* \begin{pmatrix} i \\ \ell \end{pmatrix} \Delta^* \begin{pmatrix} j \\ k \end{pmatrix}$ *holds for pairwise distict indices* i*,* j*,* k*,* $\ell = 0,\ldots, n-1$*,*

(a) $\Delta \begin{pmatrix} i \\ k \end{pmatrix} \Delta \begin{pmatrix} j \\ \ell \end{pmatrix} = \Delta \begin{pmatrix} i \\ \ell \end{pmatrix} \Delta \begin{pmatrix} j \\ k \end{pmatrix}$ *holds for pairwise distict indices* i*,* j*,* k*,* $\ell = 0,\ldots, n-1$*.*

Notice that the label (a)* was already used in Introduction. To prove the lemma above, we need the following lemma, which is similar to the case of simplices of the sphere.

Lemma 10. *We have*

$$\Delta^*(k_0 \cdots k_m) = \Delta^*(k_0 \cdots k_{m-1}) - \sum_{r=0}^{m-1} \Delta^* \begin{pmatrix} k_r \\ k_m \end{pmatrix} \Delta^* \left(k_0 \cdots \widehat{k_r} \cdots k_{m-1} \begin{matrix} k_m \\ k_r \end{matrix} \right), \quad (53)^*$$

and the equation replaced Δ^* *with* Δ *of the equation above, named* (53)*, for an integer* $m = 0,\ldots, n-1$ *and pairwise distinct indices* $k_0,\ldots, k_m$*. If* (a)* *holds, then we have*

$$\Delta^* \left(k_0 \cdots k_{m-1} \begin{matrix} i \\ j \end{matrix} \right) = \Delta^* \left(k_0 \cdots k_{m-2} \begin{matrix} i \\ j \end{matrix} \right) - \Delta^* \begin{pmatrix} i \\ k_{m-1} \end{pmatrix} \Delta^* \left(k_0 \cdots k_{m-2} \begin{matrix} k_{m-1} \\ j \end{matrix} \right), \quad (54)^*$$

$$\Delta^* \left(k_0 \cdots k_{m-1} \begin{matrix} i \\ j \end{matrix} \right) \Delta^* \left(k'_0 \cdots k'_{m'-1} \begin{matrix} i' \\ j' \end{matrix} \right) = \Delta^* \left(k_0 \cdots k_{m-1} \begin{matrix} i' \\ j \end{matrix} \right) \Delta^* \left(k'_0 \cdots k'_{m'-1} \begin{matrix} i \\ j' \end{matrix} \right), \quad (55)^*$$

$$\Delta^* \left(k_0 \cdots k_{m-1} k \begin{matrix} i \\ j \end{matrix} \right) \Delta^* \left(k'_0 \cdots k'_{m'-1} \begin{matrix} i' \\ j' \end{matrix} \right) = \Delta^* \left(k_0 \cdots k_{m-1} \begin{matrix} i \\ j \end{matrix} \right) \Delta^* \left(k'_0 \cdots k'_{m'-1} k \begin{matrix} i' \\ j' \end{matrix} \right), \quad (56)^*$$

$$\Delta^*\begin{pmatrix} k_0\cdots k_{m-1}k & i \\ & j\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m'-1} & i' \\ & j'\end{pmatrix} = \Delta^*\begin{pmatrix} k_0\cdots k_{m-1} & i' \\ & j\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m'-1}k & i \\ & j'\end{pmatrix}, \tag{57}^*$$

$$\Delta^*\begin{pmatrix} k_0\cdots k_{m-1}k & i \\ & j\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m'-1}k' & i' \\ & j'\end{pmatrix} = \Delta^*\begin{pmatrix} k_0\cdots k_{m-1}k' & i' \\ & j\end{pmatrix}\Delta^*\begin{pmatrix} k'_0\cdots k'_{m'-1}k & i \\ & j'\end{pmatrix}, \tag{58}^*$$

for integers m, $m' = 0, \ldots, n-1$ *and indices* $k_0, \ldots, k_{m-1}$, $k'_0, \ldots, k'_{m'-1}$, i, j, i', j', k, *and* k' *such that indices appearing in an identical* $\Delta^*(\cdots)$ *are pairwise distinct. If* (a) *holds, then we have equations replaced* Δ^* *with* Δ *of five equations above, named* (54), (55), (56), (57), *and* (58), *respectively.*

Proof. The proof is essentially same to Lemmas 7, 8, 9 and Corollaries 10, 11 of [7]. $(53)^*$ and $(54)^*$ are Laplace expansions. We have $(55)^*$ by induction: $(55)^*$ for (m, m') with $m \geqq m' > 0$ comes from $(55)^*$ for $(m, m'-1)$ and $(m, 0)$, and $(55)^*$ for $(m, 0)$ with $m > 0$ comes from $(55)^*$ for $(m-1, 0)$. $(56)^*$ comes from $(54)^*$ and $(55)^*$. $(57)^*$ and $(58)^*$ come from $(54)^*$–$(56)^*$ and $(54)^*$–$(57)^*$ where the proofs are divided into two cases of $k \neq i'$ and $k = i'$, and three cases of $k \neq i'$, $k' \neq i$, and $k = i'$ and $k' = i$, respectively. The proofs of (53)–(58) are similar. □

Remark that $(53)^*$ and (53) is valid with assumption neither $(a)^*$ nor (a). Lemma 10 is useful to prove the equivalence of $(a)^*$ and (a).

Proof of Lemma 9. The proof is essentially same to Lemma 12 of [7]. In particular, $(a)^*$ implies $(58)^*$, so, for pairwise distinct indices i, j, i', and j', we have

$$\begin{aligned}&\Delta^*\begin{pmatrix} 0\cdots\widehat{i}\cdots\widehat{j}\cdots n-1 & j \\ & i\end{pmatrix}\Delta^*\begin{pmatrix} 0\cdots\widehat{i'}\cdots\widehat{j'}\cdots n-1 & j' \\ & i'\end{pmatrix}\\ &\quad = \Delta^*\begin{pmatrix} 0\cdots\widehat{i}\cdots\widehat{j'}\cdots n-1 & j' \\ & i\end{pmatrix}\Delta^*\begin{pmatrix} 0\cdots\widehat{i'}\cdots\widehat{j}\cdots n-1 & j \\ & i'\end{pmatrix}.\end{aligned}$$

The equation above and $(40)^*$ imply (a): $\Delta\begin{pmatrix} i \\ j\end{pmatrix}\Delta\begin{pmatrix} i' \\ j'\end{pmatrix} = \Delta\begin{pmatrix} i' \\ j\end{pmatrix}\Delta\begin{pmatrix} i \\ j'\end{pmatrix}$. The proof of the converse (a) $\Rightarrow$ $(a)^*$ is similar. □

In order to consider orthocenters, we have to define four kinds of values μ_k^*, μ_k, ν_k^*, ν_k. To define them, we need the following lemma.

Lemma 11. *Assume* $(a)^*$ (*which is equivalent to* (a)). *Then, we have*

$$\Delta^*\begin{pmatrix} i \\ k\end{pmatrix}\Delta^*\begin{pmatrix} k \\ j\end{pmatrix}\Delta^*\begin{pmatrix} i' \\ j'\end{pmatrix} = \Delta^*\begin{pmatrix} i' \\ k\end{pmatrix}\Delta^*\begin{pmatrix} k \\ j'\end{pmatrix}\Delta^*\begin{pmatrix} i \\ j\end{pmatrix}, \tag{59}^*$$

$$\Delta^*\begin{pmatrix} i \\ k\end{pmatrix}\Delta^*\begin{pmatrix} k \\ j\end{pmatrix}\Delta^*\begin{pmatrix} k & i' \\ & j'\end{pmatrix} = \Delta^*\begin{pmatrix} i' \\ k\end{pmatrix}\Delta^*\begin{pmatrix} k \\ j'\end{pmatrix}\Delta^*\begin{pmatrix} k & i \\ & j\end{pmatrix}, \tag{60}^*$$

for indices i, j, i', j', k *such that indices appearing in an identical* $\Delta^*(\cdots)$ *are pairwise distinct. On the same assumption, we also have equations replaced* Δ^* *with* Δ *of the equations above, named* (59) *and* (60), *respectively.*

Proof. They are proved essentially same to Lemma 15 of [7]. (59)* comes from (a)* where the proof is divided into two cases of $\{i,j\}\cap\{i',j'\}\neq\emptyset$ and $\{i,j\}\cap\{i',j'\}=\emptyset$. (60)* comes from (59)*. The proofs of (59) and (60) are similar. □

Definition 2. *On the assumption* (a)* (*which is equivalent to* (a))*, for an index* $k=0,\ldots,n-1$*, if there exists a pair of distinct indices* $i,j=0,\ldots,\widehat{k},\ldots,n-1$ *such that* $\Delta^*\begin{pmatrix} i \\ j \end{pmatrix}\neq 0$ *(resp.* $-\Delta^*\begin{pmatrix} k\,i \\ j \end{pmatrix}\neq 0$, $-\Delta\begin{pmatrix} i \\ j \end{pmatrix}\neq 0$, $-\Delta\begin{pmatrix} k\,i \\ j \end{pmatrix}\neq 0$*), we define*

$$\mu_k^*=\frac{\Delta^*\begin{pmatrix} i \\ k \end{pmatrix}\Delta^*\begin{pmatrix} k \\ j \end{pmatrix}}{\Delta^*\begin{pmatrix} i \\ j \end{pmatrix}} \qquad \Big(\text{resp. } \nu_k^*=\frac{\Delta^*\begin{pmatrix} i \\ k \end{pmatrix}\Delta^*\begin{pmatrix} k \\ j \end{pmatrix}}{-\Delta^*\begin{pmatrix} k\,i \\ j \end{pmatrix}}, \tag{61}^*$$

$$\mu_k=-\frac{\left(-\Delta\begin{pmatrix} i \\ k \end{pmatrix}\right)\left(-\Delta\begin{pmatrix} k \\ j \end{pmatrix}\right)}{-\Delta\begin{pmatrix} i \\ j \end{pmatrix}}, \qquad \nu_k=\frac{\left(-\Delta\begin{pmatrix} i \\ k \end{pmatrix}\right)\left(-\Delta\begin{pmatrix} k \\ j \end{pmatrix}\right)}{-\Delta\begin{pmatrix} k\,i \\ j \end{pmatrix}}\Big) \tag{61}$$

which depends only the index $k=0,\ldots,n-1$.

Remark 4. There exists μ_k^* and it is positive for each index k, because $\Delta^*\begin{pmatrix} i \\ j \end{pmatrix}=-\langle\mathbf{p}_i^*|\mathbf{p}_j^*\rangle>1$ for arbitrary pair of distinct indices i,j from Lemma 3.

For four kinds of values above, we have the following three lemmas.

Lemma 12. *Assume* (a)*. *Then, we have*

$$(-1)^m\Delta^*\begin{pmatrix} k_0\cdots k_{m-1}\,i \\ j \end{pmatrix}=\sqrt{\mu_i^*}\sqrt{\mu_j^*}\prod_{r=0}^{m-1}(\mu_{k_r}^*-1), \tag{62}$$

for an integer $m=0,\ldots,n-2$ *and pairwise distinct indices* $k_0,\ldots,k_{m-1}$, i, *and* j,

$$(-1)^m\Delta^*(k_0\cdots k_m)=\sum_{s=0}^{m}\mu_{k_s}^*\prod_{r=0}^{r\neq s}{}^{m}(\mu_{k_r}^*-1)-\prod_{r=0}^{m}(\mu_{k_r}^*-1), \tag{63}$$

for an integer $m=-1,\ldots,n-1$ *and pairwise distinct indices* $k_0,\ldots,k_m$.

Proof. We have (62) by the following induction (it is trivial for $m=0$ and $m=1$):

$$\begin{aligned}(-1)^m\Delta^*\begin{pmatrix} k_0\cdots k_{m-1}\,i \\ j \end{pmatrix}&=\frac{-\Delta^*\begin{pmatrix} k_{m-1}\,i' \\ j' \end{pmatrix}}{\Delta^*\begin{pmatrix} i' \\ j' \end{pmatrix}}\cdot(-1)^{m-1}\Delta^*\begin{pmatrix} k_0\cdots k_{m-2}\,i \\ j \end{pmatrix}\\&=\frac{\sqrt{\mu_{i'}^*}\sqrt{\mu_{j'}^*}(\mu_{k_{m-1}}^*-1)}{\sqrt{\mu_{i'}^*}\sqrt{\mu_{j'}^*}}\cdot\sqrt{\mu_i^*}\sqrt{\mu_j^*}\prod_{r=0}^{m-2}(\mu_{k_r}^*-1)\\&=\sqrt{\mu_i^*}\sqrt{\mu_j^*}\prod_{r=0}^{m-1}(\mu_{k_r}^*-1),\end{aligned}$$

where the first equality comes from (56)* with some distinct indices i', $j' = 0, \ldots \widehat{k_{m-1}} \ldots, n-1$. We also have (63) by the following induction (for $m = -1$, we have $(-1)^{-1}\Delta^*() = 0 - 1$):

$$\begin{aligned}
(-1)^m\Delta^*(k_0\cdots k_m) &= \sum_{s=0}^{m-1}\Delta^*\begin{pmatrix}k_s\\k_m\end{pmatrix}\cdot(-1)^{m-1}\Delta^*\begin{pmatrix}k_0\cdots\widehat{k_s}\cdots k_{m-1}\,k_m\\k_s\end{pmatrix}\\
&\qquad -(-1)^{m-1}\Delta^*(k_0\cdots k_{m-1})\\
&= \sum_{s=0}^{m-1}\sqrt{\mu^*_{k_s}}\sqrt{\mu^*_{k_m}}\cdot\sqrt{\mu^*_{k_m}}\sqrt{\mu^*_{k_s}}\prod_{r=0}^{\substack{r\neq s\\ m-1}}(\mu^*_{k_r}-1)\\
&\qquad -\Big(\sum_{s=0}^{m-1}\mu^*_{k_s}\prod_{r=0}^{\substack{r\neq s\\ m-1}}(\mu^*_{k_r}-1)-\prod_{r=0}^{m-1}(\mu^*_{k_r}-1)\Big)\\
&= \Big(\sum_{s=0}^{m-1}\sqrt{\mu^*_{k_s}}\sqrt{\mu^*_{k_m}}\cdot\sqrt{\mu^*_{k_m}}\sqrt{\mu^*_{k_s}}\prod_{r=0}^{\substack{r\neq s\\ m-1}}(\mu^*_{k_r}-1)\\
&\qquad -\sum_{s=0}^{m-1}\mu^*_{k_s}\prod_{r=0}^{\substack{r\neq s\\ m-1}}(\mu^*_{k_r}-1)+\mu^*_{k_m}\prod_{r=0}^{m-1}(\mu^*_{k_r}-1)\Big)\\
&\qquad -(\mu^*_{k_m}-1)\prod_{r=0}^{m-1}(\mu^*_{k_r}-1)\\
&= \sum_{s=0}^{m}\mu^*_{k_s}\prod_{r=0}^{\substack{r\neq s\\ m}}(\mu^*_{k_r}-1)-\prod_{r=0}^{m}(\mu^*_{k_r}-1),
\end{aligned}$$

where the first equality comes from (53)*. □

Lemma 13. *If* (a)* *holds, for each index* $k = 0,\ldots, n-1$, *there exists* ν_k *and we have*

$$1-\nu_k = \frac{\mu^*_k - 1}{\langle \mathbf{p}^*_k|\mathbf{p}_k\rangle^2}. \tag{64}$$

Proof. Assume that there does not exist ν_k for some index k. Then, for arbitrary pair of distinct indices i, $j = 0,\ldots \widehat{k} \ldots, n-1$,

$$\begin{aligned}
-\Delta\begin{pmatrix}k\,i\\j\end{pmatrix} &= \frac{(-1)^{n-3}\Delta^*\begin{pmatrix}0\cdots\widehat{i}\cdots\widehat{j}\cdots\widehat{k}\cdots n-1\,j\\i\end{pmatrix}}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\langle\mathbf{p}^*_i|\mathbf{p}_i\rangle\langle\mathbf{p}^*_j|\mathbf{p}_j\rangle\langle\mathbf{p}^*_k|\mathbf{p}_k\rangle^2\\
&= \frac{\sqrt{\mu^*_j}\sqrt{\mu^*_i}\prod_{\ell=0}^{\substack{\ell\neq i,j,k\\ n-1}}(\mu^*_\ell-1)}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\langle\mathbf{p}^*_i|\mathbf{p}_i\rangle\langle\mathbf{p}^*_j|\mathbf{p}_j\rangle\langle\mathbf{p}^*_k|\mathbf{p}_k\rangle^2
\end{aligned} \tag{65}$$

is equal to 0, which means that there exists $\ell = 0,\ldots\widehat{i}\ldots\widehat{j}\ldots\widehat{k}\ldots, n-1$ such that $\mu^*_\ell = 1$. The indices i and j are arbitrary, so, there exist three indices ℓ, ℓ',

$\ell'' = 0,\ldots\widehat{k}\ldots, n-1$ such that all of

$$\mu^*_\ell = \frac{\Delta^*\begin{pmatrix}\ell''\\ \ell\end{pmatrix}\Delta^*\begin{pmatrix}\ell\\ \ell'\end{pmatrix}}{\Delta^*\begin{pmatrix}\ell'\\ \ell''\end{pmatrix}}, \quad \mu^*_{\ell'} = \frac{\Delta^*\begin{pmatrix}\ell\\ \ell'\end{pmatrix}\Delta^*\begin{pmatrix}\ell'\\ \ell''\end{pmatrix}}{\Delta^*\begin{pmatrix}\ell''\\ \ell\end{pmatrix}}, \quad \mu^*_{\ell''} = \frac{\Delta^*\begin{pmatrix}\ell'\\ \ell''\end{pmatrix}\Delta^*\begin{pmatrix}\ell''\\ \ell\end{pmatrix}}{\Delta^*\begin{pmatrix}\ell\\ \ell'\end{pmatrix}} \tag{66}$$

are equal to 1. This is a contradiction to $\Delta^*\begin{pmatrix}\ell'\\ \ell''\end{pmatrix}$, $\Delta^*\begin{pmatrix}\ell''\\ \ell\end{pmatrix}$, $\Delta^*\begin{pmatrix}\ell\\ \ell'\end{pmatrix} > 1$. Hence, for each k, there exists ν_k, so, there exist distinct indices $i, j = 0,\ldots\widehat{k}\ldots, n-1$ such that $-\Delta\begin{pmatrix}k\, i\\ \quad j\end{pmatrix} \neq 0$. Similarly to (65), we have

$$-\Delta\begin{pmatrix}i\\ j\end{pmatrix} = \frac{\sqrt{\mu^*_j}\sqrt{\mu^*_i}\prod\limits_{\ell=0}^{n-1}{}^{\ell\neq i,j}(\mu^*_\ell - 1)}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\langle \mathbf{p}^*_i|\mathbf{p}_i\rangle\langle \mathbf{p}^*_j|\mathbf{p}_j\rangle. \tag{67}$$

Two equations (65) and (67) imply

$$1-\nu_k = \frac{-\Delta\begin{pmatrix}i\\ j\end{pmatrix}}{-\Delta\begin{pmatrix}k\, i\\ \quad j\end{pmatrix}} = \frac{\mu^*_k - 1}{\langle \mathbf{p}^*_k|\mathbf{p}_k\rangle^2}.$$

□

Lemma 14. *Assume* (a)*. *Then, we have*

$$\left(-\Delta\begin{pmatrix}i\\ j\end{pmatrix}\right)^2(1-\nu_i)(1-\nu_j) = \nu_i\nu_j, \tag{68}$$

for distinct indices i and j,

$$-\Delta\begin{pmatrix}k_0\cdots k_{m-1}\, i\\ \qquad\qquad j\end{pmatrix}\prod_{r=0}^{m-1}(1-\nu_{k_r}) = -\Delta\begin{pmatrix}i\\ j\end{pmatrix}, \tag{69}$$

for an integer $m = 0,\ldots, n-2$ and pairwise distinct indices $k_0,\ldots, k_{m-1}$, i, and j,

$$\Delta(k_0\cdots k_m)\prod_{r=0}^{m}(1-\nu_{k_r}) = 1 - \sum_{r=0}^{m}\nu_{k_r}, \tag{70}$$

for an integer $m = -1,\ldots, n-1$ and pairwise distinct indices $k_0,\ldots, k_m$.

Proof. For (68), let k, ℓ, k', and ℓ' be indices such that $k \neq \ell \neq i \neq k \neq j$, $k' \neq \ell' \neq j \neq k' \neq i$, and

$$-\Delta\begin{pmatrix}i\, k\\ \ell\end{pmatrix} \neq 0 \neq -\Delta\begin{pmatrix}j\, k'\\ \ell'\end{pmatrix}.$$

Then we have

$$\left(-\Delta\begin{pmatrix}i\\ j\end{pmatrix}\right)(1-\nu_i)\cdot\left(-\Delta\begin{pmatrix}j\\ i\end{pmatrix}\right)(1-\nu_j)$$

$$
\begin{aligned}
&= \left(-\Delta\binom{i}{j}\right)\frac{-\Delta\binom{k}{\ell}}{-\Delta\binom{i\,k}{j\,\ell}}\cdot\left(-\Delta\binom{j}{i}\right)\frac{-\Delta\binom{k'}{\ell'}}{-\Delta\binom{j\,k'}{i\,\ell'}}\\
&= \frac{\left(-\Delta\binom{i}{\ell}\right)\left(-\Delta\binom{k}{j}\right)}{-\Delta\binom{i\,k}{j\,\ell}}\cdot\frac{\left(-\Delta\binom{j}{\ell'}\right)\left(-\Delta\binom{k'}{i}\right)}{-\Delta\binom{j\,k'}{i\,\ell'}}\\
&= \frac{\left(-\Delta\binom{i}{\ell}\right)\left(-\Delta\binom{k}{i}\right)}{-\Delta\binom{i\,k}{j\,\ell}}\cdot\frac{\left(-\Delta\binom{j}{\ell'}\right)\left(-\Delta\binom{k'}{j}\right)}{-\Delta\binom{j\,k'}{i\,\ell'}}\\
&= \nu_i\cdot\nu_j.
\end{aligned}
$$

The equation (69) comes from

$$
\begin{aligned}
-\Delta\binom{k_0\cdots k_{\ell-1}k\,i}{j}(1-\nu_k) &= -\Delta\binom{k_0\cdots k_{\ell-1}k\,i}{j}\frac{-\Delta\binom{i'}{j'}}{-\Delta\binom{k\,i'}{j'}}\\
&= -\Delta\binom{k_0\cdots k_{\ell-1}\,i}{j}, \qquad (71)
\end{aligned}
$$

where the last equality comes from (56) with some distinct indices $i', j' = 0,\ldots\widehat{k}\ldots, n-1$. We also have (70) by

$$
\begin{aligned}
&\Delta(k_0\cdots k_m)\prod_{r=0}^{m}(1-\nu_{k_r})\\
&= \Big(\Delta(k_0\cdots k_{m-1})\prod_{r=0}^{m-1}(1-\nu_{k_r})\\
&\quad -\sum_{\ell=0}^{m-1}\left(-\Delta\binom{k_\ell}{k_m}\right)\left(-\Delta\binom{k_0\cdots\widehat{k_\ell}\cdots k_{m-1}\,k_m}{k_\ell}\prod_{r=0}^{m-1}{}^{r\neq\ell}(1-\nu_{k_r})\right)(1-\nu_{k_\ell})\Big)(1-\nu_{k_m})\\
&= \Big((1-\sum_{r=0}^{m-1}\nu_{k_r})-\sum_{\ell=0}^{m-1}\left(-\Delta\binom{k_\ell}{k_m}\right)\left(-\Delta\binom{k_m}{k_\ell}\right)(1-\nu_{k_\ell})\Big)(1-\nu_{k_m})\\
&= (1-\sum_{r=0}^{m-1}\nu_{k_r})(1-\nu_{k_m})-\sum_{\ell=0}^{m-1}\nu_{k_\ell}\nu_{k_m}\\
&= 1-\sum_{r=0}^{m}\nu_{k_r},
\end{aligned}
$$

where the first equality comes from (53), the second equality comes from the assumption of induction and (69), and the third equality comes from (68). □

If there exists ν_k^* (resp. μ_k) for an index $k = 0,\ldots, n-1$, it holds that

$$(\mu_k^* - 1)(\nu_k^* - 1) = \frac{-\Delta^*\begin{pmatrix} k & i \\ & j \end{pmatrix}}{\Delta^*\begin{pmatrix} i \\ j \end{pmatrix}} \cdot \frac{\Delta^*\begin{pmatrix} i' \\ j' \end{pmatrix}}{-\Delta^*\begin{pmatrix} k & i' \\ & j' \end{pmatrix}} = 1 \tag{72}^*$$

$$\text{(resp.}\quad (1-\mu_k)(1-\nu_k) = \frac{-\Delta\begin{pmatrix} k & i \\ & j \end{pmatrix}}{-\Delta\begin{pmatrix} i \\ j \end{pmatrix}} \cdot \frac{-\Delta\begin{pmatrix} i' \\ j' \end{pmatrix}}{-\Delta\begin{pmatrix} k & i' \\ & j' \end{pmatrix}} = 1), \tag{72}$$

with some pairs of distinct indices i, j; i', $j' = 0,\ldots \widehat{k} \ldots, n-1$. The following two lemmas are natural consequences of (64), (72)*, and (72).

Lemma 15. *If* (a)* *holds, for each index* $k = 0,\ldots, n-1$, *the followings are equivalent:*

(d)* $\mu_k^* \neq 1$,
(e)* *there exists* ν_k^*,
(d) *there exists* μ_k,
(e) $\nu_k \neq 1$.

Proof. The equivalence of (e) and (d)* comes from (64). The negation of (e)* means that

$$-\Delta^*\begin{pmatrix} k & i \\ & j \end{pmatrix} = \Delta^*\begin{pmatrix} i \\ k \end{pmatrix}\Delta^*\begin{pmatrix} k \\ j \end{pmatrix} - \Delta^*\begin{pmatrix} i \\ j \end{pmatrix}$$

is equal to 0 for arbitrary distinct i, $j = 0,\ldots \widehat{k} \ldots, n-1$, which means $\mu_k^* = 1$. So (e)* and (d)* are equivalent. The negation of (d) means that (67) is equal to 0 for arbitrary distinct i, $j = 0,\ldots \widehat{k} \ldots, n-1$, which means the existence of $\ell = 0,\ldots \widehat{i} \ldots \widehat{j} \ldots, n-1$ such that $\mu_\ell^* = 1$. It is equivalent to

$\mu_k^* = 1$ or the existence of three indices $\ell, \ell', \ell'' \neq k$ such that $\mu_\ell^* = \mu_{\ell'}^* = \mu_{\ell''}^* = 1$.

However, the latter condition is impossible (see around (66) in the proof of Lemma 13), so it is equivalent to $\mu_k^* = 1$, that is, (d) is equivalent to (d)*. □

Corollary 16. *If* (a)* *holds, there exists at most one index* $i = 0,\ldots, n-1$ *such that* μ_i (*resp.* ν_i^*) *does not exist.*

Proof. Assume the nonentity of μ_i (resp. ν_i^*) for an index i. Then it implies $\nu_i = 1$, so, from (68), we have $\nu_j = 0 \neq 1$ for each $j = 0,\ldots \widehat{i} \ldots, n-1$, which means the existence of μ_j (resp. ν_j^*). □

Lemma 17. *On the assumption* (a)*, *we have*

$$1 - \mu_k = (\nu_k^* - 1)\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle^2, \tag{64}^*$$

for each index $k = 0,\ldots, n-1$ *such that there exists* ν_k^* (*the existence of such* k *comes from Corollary* 16).

Proof. Neither μ_k^* nor ν_k is equal to 1 from Lemma 15, so, (64)* comes from (64), (72)*, and (72). □

5. Explicit representation of the orthocenter.

To represent the orthocenter, we need the following vector.

Lemma 18. *If* (a)* *holds, the vector*

$$\tilde{\mathbf{h}} = \frac{1}{\sqrt{\mu_k^*}}\left(\mathbf{p}_k^* - \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle}\mathbf{p}_k\right) = \frac{1}{\sqrt{\mu_k^*}}\left(\mathbf{p}_k^* - (1-\nu_k)\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle \mathbf{p}_k\right) \in \mathbb{R}^n \setminus \{\mathbf{0}\} \quad (73)$$

does not depend on $k = 0, \ldots, n-1$.

Proof. To show

$$\frac{1}{\sqrt{\mu_k^*}}(\mathbf{p}_k^* - \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle}\mathbf{p}_k) = \frac{1}{\sqrt{\mu_\ell^*}}(\mathbf{p}_\ell^* - \frac{\mu_\ell^* - 1}{\langle \mathbf{p}_\ell^* | \mathbf{p}_\ell \rangle}\mathbf{p}_\ell)$$

for distinct indices k and ℓ, it is enough to compare the pseudo-inner products $\langle \mathbf{p}_i^* | \cdot \rangle$ of both sides for all indices $i = 0, \ldots, n-1$. □

Lemma 19. *If* (a)* *holds, we have*

$$\langle \tilde{\mathbf{h}} | \mathbf{p}_k^* \rangle = -\sqrt{\mu_k^*}, \quad (74)$$

$$\langle \tilde{\mathbf{h}} | \mathbf{p}_k \rangle = \frac{\nu_k}{\sqrt{\mu_k^*}} \langle \mathbf{p}_k^* | \mathbf{p}_k \rangle, \quad (75)$$

for each index k,

$$\langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \tilde{\mathbf{h}} \rangle = \langle \mathbf{p}_i^* | \tilde{\mathbf{h}} \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle, \quad (76)$$

for distinct indices i, j, *and* k, *and,*

$$-\langle \tilde{\mathbf{h}} | \tilde{\mathbf{h}} \rangle = \frac{\sum_{j=0}^{n-1} \mu_j^* \prod_{i=0}^{i \neq j}{}^{n-1} (\mu_i^* - 1)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} = \sum_{j=0}^{n-1} \nu_j. \quad (77)$$

Proof. (74), (75), and (76) are obvious. The former equality of (77) comes from

$$\begin{aligned}
-\langle \tilde{\mathbf{h}} | \tilde{\mathbf{h}} \rangle &= \frac{-1}{\sqrt{\mu_k^*}\sqrt{\mu_\ell^*}} \left\langle \mathbf{p}_k^* - \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle}\mathbf{p}_k \middle| \mathbf{p}_\ell^* - \frac{\mu_\ell^* - 1}{\langle \mathbf{p}_\ell^* | \mathbf{p}_\ell \rangle}\mathbf{p}_\ell \right\rangle \\
&= \frac{-1}{\sqrt{\mu_k^*}\sqrt{\mu_\ell^*}} \left(-\Delta^* \binom{k}{\ell} + \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle} \frac{\mu_\ell^* - 1}{\langle \mathbf{p}_\ell^* | \mathbf{p}_\ell \rangle} \Delta \binom{k}{\ell} \right) \\
&= 1 + \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle} \frac{\mu_\ell^* - 1}{\langle \mathbf{p}_\ell^* | \mathbf{p}_\ell \rangle} \frac{-\Delta \binom{k}{\ell}}{\sqrt{\mu_k^*}\sqrt{\mu_\ell^*}} \\
&= 1 + \frac{\prod_{i=0}^{n-1} (\mu_i^* - 1)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \\
&= \frac{\sum_{j=0}^{n-1} \mu_j^* \prod_{i=0}^{i \neq j}{}^{n-1} (\mu_i^* - 1)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)},
\end{aligned} \quad (78)$$

where k and ℓ are some distinct indices and the fourth and last equalities come from (67) and (63), respectively. The latter equality of (77) comes from

$$\begin{aligned}
&\frac{\mu_j^* \prod_{i=0}^{n-1}{}^{i\neq j}(\mu_i^* - 1)}{(-1)^{n-1}\Delta^*(0\cdots n-1)} \\
&= -\Delta(0\cdots n-1)\frac{\mu_j^*}{\langle \mathbf{p}_j^*|\mathbf{p}_j\rangle^2}\prod_{i=0}^{n-1}{}^{i\neq j}\frac{\mu_i^*-1}{\langle \mathbf{p}_i^*|\mathbf{p}_i\rangle^2} \\
&= -\Delta(0\cdots n-1)\Big((1-\nu_j)+\frac{1}{\langle \mathbf{p}_j^*|\mathbf{p}_j\rangle^2}\Big)\prod_{i=0}^{n-1}{}^{i\neq j}(1-\nu_i) \\
&= -\Delta(0\cdots n-1)\prod_{i=0}^{n-1}(1-\nu_i) + \Delta(0\cdots \widehat{j}\cdots n-1)\prod_{i=0}^{n-1}{}^{i\neq j}(1-\nu_i) \\
&= (\sum_{i=0}^{n-1}\nu_i - 1) + (1-\sum_{i=0}^{n-1}{}^{i\neq j}\nu_i) \\
&= \nu_j,
\end{aligned} \tag{79}$$

where the first, second, third, and fourth equalities come from (39), (64), (34), and (70), respectively. □

The following two theorems are main results.

Theorem 20. *If* (a)* *and* $\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle < 0$ *holds, then*

$$\mathbf{h} = \frac{\tilde{\mathbf{h}}}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} \tag{80}$$

is in $\mathbb{H}^{n-1}$ *and it is the orthocenter of* S.

Proof. It is obvious that $\mathbf{h} \in \mathbb{H}^{n-1}$ or $-\mathbf{h} \in \mathbb{H}^{n-1}$. If $-\mathbf{h} \in \mathbb{H}^{n-1}$ then, for each k, $\langle\mathbf{h}|\mathbf{p}_k^*\rangle = -\langle-\mathbf{h}|\mathbf{p}_k^*\rangle \geqq 1$ from Lemma 3, which is a contradiction to

$$\langle\mathbf{h}|\mathbf{p}_k^*\rangle = \frac{\langle\tilde{\mathbf{h}}|\mathbf{p}_k^*\rangle}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} = \frac{-\sqrt{\mu_k^*}}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} < 0.$$

From (73) and Definition 1, $\mathbf{h}$ is the orthocenter. □

Remark 5. For an index $k = 0,\ldots, n-1$ such that there exists μ_k, let

$$\tilde{\mathbf{h}}_k^* = \mathbf{p}_k - \frac{1-\mu_k}{\langle\mathbf{p}_k^*|\mathbf{p}_k\rangle}\mathbf{p}_k^* = \mathbf{p}_k - (\nu_k^* - 1)\langle\mathbf{p}_k^*|\mathbf{p}_k\rangle\mathbf{p}_k^* \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{81}$$

Then we have

$$\tilde{\mathbf{h}}_k^* = \frac{-(1-\mu_k)\sqrt{\mu_k^*}}{\langle\mathbf{p}_k^*|\mathbf{p}_k\rangle}\tilde{\mathbf{h}}, \qquad \Delta\binom{k}{i}\tilde{\mathbf{h}}_\ell^* = \Delta\binom{\ell}{i}\tilde{\mathbf{h}}_k^*, \qquad \mu_k\tilde{\mathbf{h}}_\ell^* = \Delta\binom{\ell}{k}\tilde{\mathbf{h}}_k^*, \tag{82}$$

for k, ℓ, and i are pairwise distinct indices such that there exist μ_k and μ_ℓ. In particular, we have the first equation directly:

$$\tilde{\mathbf{h}}_k^* = \mathbf{p}_k - \frac{1-\mu_k}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle} \mathbf{p}_k^* = \frac{1-\mu_k}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle} \left(\frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle} \mathbf{p}_k - \mathbf{p}_k^* \right) = \frac{-(1-\mu_k)\sqrt{\mu_k^*}}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle} \tilde{\mathbf{h}},$$

and we also have the second and third equations by comparing the pseudo-inner products of both sides and $\mathbf{p}_j$ for each index $j = 0,\ldots, n-1$.

Theorem 21. *The followings are equivalent:*

(b) *there exists the orthocenter,*

(c)* (a)* *and* $\displaystyle\sum_{j=0}^{n-1} \mu_j^* \prod_{\substack{i=0 \\ i\neq j}}^{n-1} (\mu_i^* - 1) > 0$ *hold,*

(c) (a) *and* $\displaystyle\sum_{j=0}^{n-1} \nu_j > 0$ *hold.*

Proof. The equivalence (c)* $\Leftrightarrow$ (c) comes from (31)*, (77), and Lemma 9. The implication (c) $\Rightarrow$ (b) comes from (77) and Theorem 20. For the implication (b) $\Rightarrow$ (c)*, assume that there exists the orthocenter $\mathbf{n}$. Then we can represent $\mathbf{n} = A_k^* \mathbf{p}_k^* + A_k \mathbf{p}_k$ with some A_k^*, $A_k \in \mathbb{R}$ for each k, so we have

$$\langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \mathbf{n} \rangle = \langle \mathbf{p}_i^* | \mathbf{n} \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle \tag{83}$$

for distinct indices i, $j = 0,\ldots \widehat{k} \ldots, n-1$, because the both sides are equal to $A_k^* \langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle$. So we also have (a)*, in particular, we have

$$\langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_\ell^* \rangle = \frac{\langle \mathbf{p}_i^* | \mathbf{n} \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle}{\langle \mathbf{p}_j^* | \mathbf{n} \rangle} \langle \mathbf{p}_j^* | \mathbf{p}_\ell^* \rangle = \frac{\langle \mathbf{p}_i^* | \mathbf{n} \rangle \langle \mathbf{p}_j^* | \mathbf{p}_\ell^* \rangle}{\langle \mathbf{p}_j^* | \mathbf{n} \rangle} \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle = \langle \mathbf{p}_i^* | \mathbf{p}_\ell^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle.$$

Moreover, (76) and (83) imply that $\langle \mathbf{p}_i^* | \tilde{\mathbf{h}} \rangle / \langle \mathbf{p}_i^* | \mathbf{n} \rangle$ does not depend on the index $i = 0,\ldots, n-1$, so $\tilde{\mathbf{h}}$ and $\mathbf{n}$ are parallel (notice that neither $\tilde{\mathbf{h}}$ nor $\mathbf{n}$ is equal to $\mathbf{0}$). Hence, $-\langle \mathbf{n} | \mathbf{n} \rangle > 0$ implies $-\langle \tilde{\mathbf{h}} | \tilde{\mathbf{h}} \rangle > 0$, which means that $\displaystyle\sum_{j=0}^{n-1} \mu_j^* \prod_{\substack{i=0 \\ i\neq j}}^{n-1} (\mu_i^* - 1) > 0$. □

6. Appendix.

Similarly to the story of the sphere [7], we consider the simplex replaced a vertex with the orthocenter of an orthocentric simplex. If we repeat this operation, a new vertex does not appear except original vertices and the orthocenter of the original simplex. We consider the generic case, i.e., there exists the orthocenter, neither $\Delta^* \begin{pmatrix} k & i \\ & j \end{pmatrix}$, $\Delta \begin{pmatrix} i \\ j \end{pmatrix}$, nor $\Delta \begin{pmatrix} k & i \\ & j \end{pmatrix}$ is equal to 0 for an arbitrary trio of pairwise disjoint indices i, j, and k, all of μ_k^*, ν_k^*, μ_k, and ν_k exist, neither of them is equal to 0, and neither of them is equal to 1 for an arbitrary index k (notice that $\Delta^* \begin{pmatrix} i \\ j \end{pmatrix}$ cannot be equal to 0. See Remark 4).

Let $S' = \mathcal{M}(S)$ be the simplex replaced the last vertex with the orthocenter of an orthocentric simplex S, i.e.,

$$\mathbf{p}'^{*}_{k} = \mathbf{p}^{*}_{k} \quad \text{for} \quad k = 0, \dots, n-2, \quad \mathbf{p}'^{*}_{n-1} = \mathbf{h}. \tag{84}$$

Then, it holds that

$$\mathbf{p}'_{k} = \frac{\mathbf{p}_k - \frac{\Delta\binom{k}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1}}{\sqrt{-\mu_k(\frac{1}{\nu_k} + \frac{1}{\nu_{n-1}})}}, \qquad \mathbf{p}'_{n-1} = \operatorname{sgn}\nu_{n-1} \cdot \mathbf{p}_{n-1}. \tag{85}$$

In particular, the pseudo-inner product

$$\Big\langle \mathbf{p}_k - \frac{\Delta\binom{k}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1} \Big| \mathbf{p}_k - \frac{\Delta\binom{k}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1} \Big\rangle = -\mu_k(\frac{1}{\nu_k} + \frac{1}{\nu_{n-1}})$$

is positive from Corollary 4 and

$$\begin{aligned}
&\Big\langle \mathbf{h} \Big| \mathbf{p}_k - \frac{\Delta\binom{k}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1} \Big\rangle \\
&= \Big\langle \frac{\tilde{\mathbf{h}}}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} \Big| \mathbf{p}_k - \frac{\Delta\binom{k}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1} \Big\rangle \\
&= \Big\langle \frac{\mathbf{p}^{*}_{n-1} - (1-\nu_{n-1})\langle\mathbf{p}^{*}_{n-1}|\mathbf{p}_{n-1}\rangle\mathbf{p}_{n-1}}{\sqrt{\mu^{*}_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} \Big| \mathbf{p}_k - \frac{\Delta\binom{k}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1} \Big\rangle \\
&= \frac{\langle\mathbf{p}^{*}_{n-1}|\mathbf{p}_{n-1}\rangle\Delta\binom{k}{n-1}}{\sqrt{\mu^{*}_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} \Big((0 - \frac{1}{\mu_{n-1}}) - (1-\nu_{n-1})(1 - \frac{1}{\mu_{n-1}}) \Big) \\
&= 0.
\end{aligned}$$

So we have the former equation of (85). We also have the latter equation of (85) from

$$\langle\mathbf{p}'^{*}_{n-1}|\operatorname{sgn}\nu_{n-1} \cdot \mathbf{p}_{n-1}\rangle = \frac{\operatorname{sgn}\nu_{n-1}}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\langle\tilde{\mathbf{h}}|\mathbf{p}_{n-1}\rangle = \frac{|\nu_{n-1}|\langle\mathbf{p}^{*}_{n-1}|\mathbf{p}_{n-1}\rangle}{\sqrt{\mu^{*}_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} > 0,$$

where the second equality comes from (75). From (76), we can define μ'^{*}_{k}, ν'^{*}_{k}, μ'_{k}, and ν'_{k} for each $k = 0,\ldots, n-1$. Similar to the Appendix of [7], we have

$$\nu'_{k} = \frac{-\nu_k}{\nu_{n-1}}, k = 0, \dots, n-2 \quad \text{and} \quad \nu'_{n-1} = \frac{1}{\nu_{n-1}}, \tag{86}$$

from

$$\Delta'\binom{i}{j} = \frac{-\Delta\begin{pmatrix} i \\ j \end{pmatrix}/\nu_{n-1}}{\sqrt{-\mu_i(\frac{1}{\nu_i}+\frac{1}{\nu_{n-1}})}\sqrt{-\mu_j(\frac{1}{\nu_j}+\frac{1}{\nu_{n-1}})}}$$

and

$$\Delta'\binom{i}{n-1} = \frac{\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix}/|\nu_{n-1}|}{\sqrt{-\mu_i(\frac{1}{\nu_i}+\frac{1}{\nu_{n-1}})}}$$

for pairwise distinct $i, j = 0,\ldots, n-2$. On the other hand, it is obvious that

$$\nu'^*_k = \nu^*_k \quad \text{for} \quad k = 0, \ldots, n-2. \tag{87}$$

We also have

$$\nu'^*_{n-1} = 1 + \frac{1}{\mu'^*_{n-1} - 1} = 1 - \sum_{k=0}^{n-1} \nu^*_k, \tag{88}$$

whose latter equality comes from

$$\begin{aligned}
\mu'^*_{n-1} &= \Delta'^*\binom{i}{n-1}\Delta'^*\binom{n-1}{j}/\Delta'^*\binom{i}{j} \\
&= \frac{\Delta^*\begin{pmatrix} i \\ n-1 \end{pmatrix}}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} \frac{\Delta^*\begin{pmatrix} n-1 \\ j \end{pmatrix}}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}/\Delta^*\begin{pmatrix} i \\ j \end{pmatrix} \\
&= \frac{1}{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle} = \frac{(-1)^{n-1}\Delta^*(0\cdots n-1)}{\prod_{\ell\neq k}^{n-1}\sum_{k=0}^{n-1}\mu^*_k\prod_{\ell=0}^{n-1}(\mu^*_\ell - 1)} \\
&= 1 - \frac{\prod_{\ell=0}^{n-1}(\mu^*_\ell - 1)}{\prod_{\ell\neq k}^{n-1}\sum_{k=0}^{n-1}\mu^*_k\prod_{\ell=0}^{n-1}(\mu^*_\ell - 1)} \\
&= 1 - \frac{1}{\sum_{k=0}^{n-1}\nu^*_k},
\end{aligned}$$

where i and j are arbitrary distinct indices of $0,\ldots, n-2$, and the second equality comes from

$$\Delta'^*\binom{i}{n-1} = \frac{-\left\langle \mathbf{p}^*_i \middle| \mathbf{p}^*_{n-1} - \frac{\mu^*_{n-1}-1}{\langle\mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle}\mathbf{p}_{n-1}\right\rangle}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} = \frac{\Delta^*\begin{pmatrix} i \\ n-1 \end{pmatrix}}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}.$$

S' is also orthocentric from (76) and

$$\begin{aligned}\sum_{j=0}^{n-1}\mu_j'^*\prod_{i=0}^{\substack{i\neq j\\ n-1}}(\mu_i'^*-1) &= \Big(\prod_{i=0}^{n-1}(\mu_i'^*-1)\Big)\cdot\sum_{j=0}^{n-1}\nu_j'^*\\ &= \Big(\prod_{i=0}^{n-2}(\mu_i^*-1)\Big)\frac{-1}{\sum\limits_{j=0}^{n-1}\nu_j^*}\cdot\Big(\sum_{j=0}^{n-2}\nu_j^*+(1-\sum_{j=0}^{n-1}\nu_j^*)\Big)\\ &= \Big(\prod_{i=0}^{n-2}(\mu_i^*-1)\Big)\frac{-1}{\sum\limits_{j=0}^{n-1}\nu_j^*}\cdot\frac{-1}{\mu_{n-1}^*-1}>0,\end{aligned}$$

where the last inequality comes from

$$\Big(\prod_{i=0}^{n-1}(\mu_i^*-1)\Big)\cdot\sum_{j=0}^{n-1}\nu_j^*=\sum_{j=0}^{n-1}\mu_j^*\prod_{i=0}^{\substack{i\neq j\\ n-1}}(\mu_i^*-1)>0.$$

The calculations above are useful to consider the following two cases of changing twice a vertex of an orthocentric simplex to the orthocenter. The former case is $S'=\mathcal{M}(S)$ and $S''=\mathcal{M}(S')$. Then the vertices of S'' are

$$\mathbf{p}_k''^*=\mathbf{p}_k'^*=\mathbf{p}_k^*\quad\text{for}\quad k=0,\dots,n-2,\quad \mathbf{p}_{n-1}''^*=\mathbf{h}'=\mathbf{p}_{n-1}^*. \tag{89}$$

In particular, $\mathbf{h}'=\mathbf{p}_{n-1}^*$ comes from

$$\begin{aligned}\sqrt{\mu_{n-1}'^*}\tilde{\mathbf{h}}' &= \mathbf{p}_{n-1}'^*-(1-\nu_{n-1}')\langle\mathbf{p}_{n-1}'^*|\mathbf{p}_{n-1}'\rangle\mathbf{p}_{n-1}'\\ &= \frac{\sqrt{\mu_{n-1}^*}\tilde{\mathbf{h}}-(1-\frac{1}{\nu_{n-1}})\langle\sqrt{\mu_{n-1}^*}\tilde{\mathbf{h}}|\mathbf{p}_{n-1}\rangle\mathbf{p}_{n-1}}{\sqrt{\mu_{n-1}^*}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\\ &= \frac{\sqrt{\mu_{n-1}^*}\tilde{\mathbf{h}}+(1-\nu_{n-1})\langle\mathbf{p}_{n-1}^*|\mathbf{p}_{n-1}\rangle\mathbf{p}_{n-1}}{\sqrt{\mu_{n-1}^*}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\\ &= \frac{\mathbf{p}_{n-1}^*}{\sqrt{\mu_{n-1}^*}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}},\end{aligned}$$

where the third equality comes from (75). The latter case is: $S'=\mathcal{M}(S)$,

$$\mathbf{p}_k''^*=\mathbf{p}_k'^*\quad\text{for}\quad k=0,\dots,n-3,\quad \mathbf{p}_{n-2}''^*=\mathbf{p}_{n-1}'^*,\quad \mathbf{p}_{n-1}''^*=\mathbf{p}_{n-2}'^*,$$

and $S'''=\mathcal{M}(S'')$. Then the vertices of S''' are

$$\mathbf{p}_k'''^*=\mathbf{p}_k''^*=\mathbf{p}_k'^*=\mathbf{p}_k^*,\quad \mathbf{p}_{n-2}'''^*=\mathbf{p}_{n-2}''^*=\mathbf{p}_{n-1}'^*=\mathbf{h},\quad \mathbf{p}_{n-1}'''^*=\mathbf{h}''=\mathbf{p}_{n-1}^*. \tag{90}$$

In particular, $\mathbf{h}''=\mathbf{p}_{n-1}^*$ comes from the impossibility of $\mathbf{h}''=-\mathbf{p}_{n-1}^*$ and

$$\begin{aligned}\sqrt{\mu_{n-1}''^*}\tilde{\mathbf{h}}'' &= \mathbf{p}_{n-1}''^*-\frac{\mathbf{p}_{n-1}''}{(\nu_{n-1}''^*-1)\langle\mathbf{p}_{n-1}''^*|\mathbf{p}_{n-1}''\rangle}\\ &= \mathbf{p}_{n-2}'^*-\frac{\mathbf{p}_{n-2}'}{(\nu_{n-2}'^*-1)\langle\mathbf{p}_{n-2}'^*|\mathbf{p}_{n-2}'\rangle}\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{p}'^{*}_{n-2} - \frac{\sqrt{-\mu_{n-2}(\frac{1}{\nu_{n-2}}+\frac{1}{\nu_{n-1}})}\mathbf{p}'_{n-2}}{(\nu'^{*}_{n-2}-1)\langle \mathbf{p}'^{*}_{n-2}|\sqrt{-\mu_{n-2}(\frac{1}{\nu_{n-2}}+\frac{1}{\nu_{n-1}})}\mathbf{p}'_{n-2}\rangle}\\
&= \mathbf{p}^{*}_{n-2} - \frac{\mathbf{p}_{n-2} - \frac{\Delta\binom{n-2}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1}}{(\nu^{*}_{n-2}-1)\langle \mathbf{p}^{*}_{n-2}|\mathbf{p}_{n-2}\rangle}\\
&= \frac{-(\tilde{\mathbf{h}}^{*}_{n-2} - \frac{\Delta\binom{n-2}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1})}{(\nu^{*}_{n-2}-1)\langle \mathbf{p}^{*}_{n-2}|\mathbf{p}_{n-2}\rangle}\\
&= \frac{-\Delta\binom{n-2}{n-1}\cdot\dfrac{\tilde{\mathbf{h}}^{*}_{n-1}-\mathbf{p}_{n-1}}{\mu_{n-1}}}{(\nu^{*}_{n-2}-1)\langle \mathbf{p}^{*}_{n-2}|\mathbf{p}_{n-2}\rangle}\\
&= \frac{-\Delta\binom{n-2}{n-1}\cdot\dfrac{\mathbf{p}^{*}_{n-1}}{\nu_{n-1}\langle \mathbf{p}^{*}_{n-1}|\mathbf{p}_{n-1}\rangle}}{(\nu^{*}_{n-2}-1)\langle \mathbf{p}^{*}_{n-2}|\mathbf{p}_{n-2}\rangle},
\end{aligned}
$$

where the fifth equality comes from (81), the sixth equality comes from (82), and the last equality comes from (72) and (81).

From two cases above, if we denote $\mathbf{p}^{*}_{n} = \mathbf{h}$, then orthocenters of the simplex with vertices $\mathbf{p}^{*}_{0},\ldots\widehat{\mathbf{p}^{*}_{k}}\ldots,\mathbf{p}^{*}_{n}$ are $\mathbf{p}^{*}_{k}$, for $k = 0,\ldots,n-1$. Moreover, for $k = 0,\ldots,n$, from (87), ν^{*}_{k} does not change if S is replaced with the simplex with vertices $\mathbf{p}^{*}_{0},\ldots\widehat{\mathbf{p}^{*}_{i}}\ldots,\mathbf{p}^{*}_{n}$ for any $i = 0,\ldots\widehat{k}\ldots,n-1$. If we denote ν'^{*}_{n-1} for $S' = \mathcal{M}(S)$ by ν^{*}_{n}, we have

$$\nu^{*}_{0} + \cdots + \nu^{*}_{n-1} + \nu^{*}_{n} = 1, \tag{91}$$

from (88) (see the equations (2) of [2], (4) of [3], and (24) of [7]).

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