Properties of a Pascal Points Circle in a Quadrilateral with Perpendicular Diagonals

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Abstract. The theory of a convex quadrilateral and a circle that forms Pascal points is a new topic in Euclidean geometry. The theory deals with the properties of the Pascal points on the sides of a convex quadrilateral and with the properties of circles that form Pascal points.

In the present paper, we shall continue developing the theory, and we shall define the concept of the “Pascal points circle”.

We shall prove four theorems regarding the properties of the points of intersection of a Pascal points circle with a quadrilateral that has intersecting perpendicular diagonals.

1. Introduction: General concepts and Fundamental Theorem of the theory of a convex quadrilateral and a circle that forms Pascal points

First, we shall briefly survey the definitions of some essential concepts of the theory of a convex quadrilateral and a circle that forms Pascal points on its sides, and then we shall present this theory’s Fundamental Theorem (see [1], [2], [3]). The theory considers the situation in which \(ABCD\) is a convex quadrilateral for which there exists a circle \(\omega\) that satisfies the following two requirements:

(i) Circle \(\omega\) passes through point \(E\), the point of intersection of the diagonals, and through point \(F\), the point of intersection of the extensions of sides \(BC\) and \(AD\).

(ii) Circle \(\omega\) intersects sides \(BC\) and \(AD\) at interior points (points \(M\) and \(N\), respectively, in Figure 1).

The Fundamental Theorem of the theory holds in this case.

The Fundamental Theorem.

Let there be: a convex quadrilateral; a circle that intersects a pair of opposite sides of the quadrilateral, that passes through the point of intersection of the extensions of these sides, and that passes through the point of intersection of the diagonals. In addition, let there be four straight lines, each of which passes both through the point of intersection of the circle with a side of the quadrilateral and through the point of intersection of the circle with the extension of a diagonal. Then there holds: the straight lines intersect at two points that are located on the other pair of opposite sides of the quadrilateral.

(In Figure 2, straight lines \(h\) and \(g\) intersect at point \(P\) on side \(AB\), and straight lines \(i\) and \(j\) intersect at point \(Q\) on side \(CD\).)
The Fundamental Theorem is proven using the general Pascal’s Theorem (see [1]).

Definitions
Because the proof of the properties of the points of intersection \( P \) and \( Q \) is based on Pascal’s Theorem,

(I) points \( P \) and \( Q \) are termed \textit{Pascal points on sides }\( AB \text{ and } CD \text{ of the quadrilateral}; \)

(II) the circle that passes through points of intersection \( E \) and \( F \) and through two opposite sides is termed a \textit{circle that forms Pascal points on the sides of the quadrilateral}.

We define a new concept: Pascal points circle.

(III) We shall call a circle whose diameter is segment \( PQ \) (see Figure 3) a \textit{Pascal points circle}.

2. Properties of a quadrilateral with perpendicular intersecting diagonals, a circle that forms Pascal points, and a Pascal points circle.

Theorem 1.
Let \( ABCD \) be a quadrilateral with perpendicular diagonals in which \( E \) is the point of intersection of the diagonals and \( F \) is the point of intersection of the extensions of the sides \( BC \) and \( AD \); \( \omega_{EF} \) is the circle whose diameter is segment \( EF \). Then,

(a) circle \( \omega_{EF} \) forms Pascal points on sides \( AB \text{ and } CD \) (see Figure 4); there are an infinite number of circles that form Pascal points on sides \( AB \text{ and } CD \);

(b) for every circle, \( \omega \), that intersects sides \( BC \) and \( AD \) at points \( M \) and \( N \), respectively, and forms Pascal points \( P \) and \( Q \) on sides \( AB \text{ and } CD \), respectively, there holds:
the point of intersection, $T$, of the tangents to circle $\omega$ at points $M$ and $N$ is the middle of segment $PQ$.

![Diagram](image.png)

**Figure 4.**

**Proof.**

(a) Let us show that circle $\omega_{EF}$ intersects sides $BC$ and $AD$ at internal points. In circle $\omega_{EF}$, angle $\angle EMF$ equals $90^\circ$. Therefore, in right triangle $\triangle BCE$, segment $EM$ is an altitude to hypotenuse $BC$, and hence it follows that the foot of altitude $EM$ (point $M$ in Figure 4) is an interior point of side $BC$. Similarly, we prove that point $N$ (the base of the altitude to hypotenuse $AD$ in right triangle $\triangle ADE$) is an internal point of side $AD$.

Based on the fundamental theorem, since circle $\omega_{EF}$ intersects sides $BC$ and $AD$ at internal points, this circle necessarily forms Pascal points on sides $AB$ and $CD$. It is clear that if there is even one circle that passes through points $E$ and $F$ and also through internal points of sides $BC$ and $AD$, then there must be an infinite number of such circles. Therefore, in our case, there are an infinite number of circles that pass through points $E$ and $F$ and through internal points of sides $BC$ and $AD$. All these circles form Pascal points on sides $AB$ and $CD$.

(b) Let us employ the following property that holds true for a convex quadrilateral (whose diagonals are not necessarily perpendicular) and a circle, $\omega$, that forms Pascal points $P$ and $Q$ on sides $AB$ and $CD$.

We denote: $M$ and $N$ are the intersection points of circle $\omega$ with sides $BC$ and $AD$, respectively, and $K$ and $L$ are the intersection points of circle $\omega$ with the extensions of diagonals $BD$ and $AC$, respectively (see Figure 5).

It thus holds that the four points $P$, $Q$, $T$, and $R$ ($P$ and $Q$ are the two Pascal points, $T$ is the point of intersection of the tangents to the circle at points $M$ and
$N$, and $R$ is the point of intersection of the tangents to the circle at points $K$ and $L$) constitute a harmonic quadruple, in other words, there holds: $\frac{QT}{TP} = \frac{QR}{RP}$ (see [3, Theorem 1].)

In our case, the quadrilateral has perpendicular diagonals. Therefore, $\angle KEL = 90^\circ$ and segment $KL$ is a diameter of $\omega$. Therefore, the tangents to circle $\omega$ at points $K$ and $L$ are parallel to each other (see Figure 6). In this case, their point of intersection, $R$, is at infinity, and ratio $\frac{QR}{RP}$ equals 1. Hence it also holds that $\frac{QT}{TP} = 1$, or $QT = TP$. In other words, point $T$ is the middle of segment $PQ$. □

Theorem 2.
Let $ABCD$ be a quadrilateral with perpendicular diagonals in which $E$ is the point of intersection of the diagonals and $F$ is the point of intersection of the extensions of sides $BC$ and $AD$; $\omega$ is a circle that passes through points $E$ and $F$ and intersects sides $BC$ and $AD$ at points $M$ and $N$, respectively; $P$ and $Q$ are Pascal points formed using $\omega$ on sides $AB$ and $CD$, respectively; $\sigma_{PQ}$ is a circle whose diameter is segment $PQ$ (a Pascal points circle); $T$ is the center of circle $\sigma_{PQ}$ (see Figure 7).

Then,
(a) Sides $BC$ and $AD$ each have at least one common point with circle $\sigma_{PQ}$. In other words:
Properties of a Pascal points circle in a quadrilateral with perpendicular diagonals

(1) In the case that the center, \( O \), of circle \( \omega \) does not belong to straight lines \( BF \) and \( AF \), circle \( \sigma_{PQ} \) intersects sides \( BC \) and \( AD \) at two points each. Two of these four points of intersection are \( N \) and \( M \); the other two points are denoted as \( V \) and \( W \) (see Figure 7).

(2) When center \( O \) lies on straight line \( BF \), circle \( \sigma_{PQ} \) is tangent to side \( BC \) at point \( M \). In this case, point \( V \) coincides with \( M \).

(3) When center \( O \) lies on straight line \( AF \), circle \( \sigma_{PQ} \) is tangent to side \( AD \) at point \( N \). In this case, point \( W \) coincides with \( N \).

(b) Points \( V \), \( T \), and \( W \) lie on the same straight line. This property holds even in cases when point \( V \) coincides with point \( M \) or point \( W \) coincides with point \( N \).

(c) Circles \( \omega \) and \( \sigma_{PQ} \) are perpendicular to each other.

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**Figure 7.**

**Proof.**

(a) We shall use the method of complex numbers in plane geometry. (The principles of the method and a system of formulas that we use in the proofs appear, for example, in [6 pp. 154-181]; some isolated formulas may be found in [4], [5]). Let us choose a system of coordinates such that circle \( \omega \) is the unit circle (\( O \) is the origin, and the radius is \( OE = 1 \)). In this system, the equation of circle \( \omega \) is \( z \cdot \bar{z} = 1 \), where \( z \) is the complex coordinate of some point \( Z \) that belongs to circle \( \omega \), and \( \bar{z} \) is the conjugate of \( z \).

We denote the complex coordinates of points \( K \), \( L \), \( M \) and \( N \) by \( k \), \( l \), \( m \) and \( n \), respectively. These points are located on unit circle \( \omega \), therefore there holds: \( \bar{k} = \frac{1}{k} \), \( \bar{l} = \frac{1}{l} \), \( \bar{m} = \frac{1}{m} \), and \( \bar{n} = \frac{1}{n} \).

Point \( P \) is the point of intersection of straight lines \( KN \) and \( LM \). Let us express the complex coordinate of \( P \) (and its conjugate) using the coordinates of points \( K \), \( L \), \( M \) and \( N \).
Let \( A(a), B(b), C(c), \) and \( D(d) \) be four points on the unit circle, and let \( S(s) \) be the point of intersection of straight lines \( AB \) and \( CD \). Then the coordinate \( s \) and its conjugate \( \overline{s} \) satisfy:

\[
\overline{s} = \frac{a + b - c - d}{ab - cd} \quad \text{and} \quad s = \frac{bcd + acd - abd - abc}{cd - ab}
\]

(I)

In our case, segment \( KL \) is a diameter of circle \( \omega \). Therefore, \( k = -l \), and the expressions for \( p \) and \( \overline{p} \) are:

\[
\overline{p} = \frac{n + k - m - l}{nk - ml} = \frac{2l + m - n}{l(m + n)} \quad \text{and} \quad p = \frac{2mn + nl - ml}{m + n}.
\]

Now, let us find complex coordinate \( t \) of point \( T \), which is the point of intersection of the tangents to the unit circle at points \( M \) and \( N \). We use the following formula:

Let \( S(s) \) be the point of intersection of the tangents to the unit circle at points \( A(a) \) and \( B(b) \), which are located on the circle. Then coordinate \( s \) and its conjugate \( \overline{s} \) satisfy:

\[
s = \frac{2ab}{a + b} \quad \text{and} \quad \overline{s} = \frac{2}{a + b}.
\]

(II)

In our case, we obtain for coordinate \( t \) and its conjugate \( \overline{t} \) the following:

\[
t = \frac{2mn}{m + n} \quad \text{and} \quad \overline{t} = \frac{2}{m + n}.
\]

Point \( T \) is the center of circle \( \sigma_{PQ} \). Therefore, the equation of circle \( \sigma_{PQ} \) is

\[
(z - t) (\overline{z} - \overline{t}) = r_{\sigma_{PQ}}^2,
\]

where \( r_{\sigma_{PQ}} \) is the radius of the circle, and \( z \) is the complex coordinate of some point \( Z \) that belongs to the circle.

Let us find the square of the radius of circle \( \sigma_{PQ} \). Point \( P \) lies on the circle, therefore the following equality holds: \( (p - t) (\overline{p} - \overline{t}) = r_{\sigma_{PQ}}^2 \).

Let us substitute the expressions for \( p \) and \( \overline{p} \) in the left-hand side of the equality. We obtain:

\[
\left( \frac{2mn + nl - ml}{m + n} - \frac{2mn}{m + n} \right) \left( \frac{2l + m - n}{l(m + n)} - \frac{2}{m + n} \right)
\]

\[
= \frac{nl - ml}{m + n} \cdot \frac{m - n}{l(m + n)} = - \left( \frac{m - n}{m + n} \right)^2.
\]

In other words, there holds \( r_{\sigma_{PQ}}^2 = - \left( \frac{m - n}{m + n} \right)^2 \). Therefore, the equation for circle \( \sigma_{PQ} \) is

\[
(z - \frac{2mn}{m + n}) \left( \overline{z} - \frac{2}{m + n} \right) = - \left( \frac{m - n}{m + n} \right)^2.
\]

(1)

Now let us find the equations of straight lines \( BC \) and \( AD \) and, subsequently, their points of intersection with circle \( \sigma_{PQ} \).
Properties of a Pascal points circle in a quadrilateral with perpendicular diagonals

We use the following formula of a straight line passing through two points \(A(a)\) and \(B(b)\) belonging to the unit circle:

\[
z + ab\bar{z} = a + b. \tag{III}
\]

In accordance with this formula, the equation of straight line \(BC\) (which passes through points \(F(f)\) and \(M(m)\) that belong to the unit circle) shall be \(z + fm\bar{z} = f + m\). Hence:

\[
\bar{z} = -\frac{1}{fm}z + \frac{f + m}{fm}. \tag{2}
\]

We substitute the expression for \(z\) from (2) into (1) and obtain:

\[
\left(\frac{z - \frac{2mn}{m + n}}{m + n}\right)\left(-\frac{1}{fm}z + \frac{f + m}{fm} - \frac{2}{m + n}\right) + \left(\frac{m - n}{m + n}\right)^2 = 0,
\]

\[
-\frac{1}{fm}z^2 + \left(\frac{f + m}{fm} - \frac{2}{m + n} + \frac{2mn}{fm(m + n)}\right)z
\]

\[
-\frac{2mn(f + m)}{fm(m + n)} + \frac{4mn}{(m + n)^2} + \frac{(m - n)^2}{(m + n)^2} = 0.
\]

This leads to the following quadratic equation:

\[
(m + n)z^2 - (3mn - fm + fn + m^2)z + fmn + 2m^2n - fm^2 = 0. \tag{3}
\]

The solutions of this equation are:

\[
z_{1,2} = \frac{3mn - fm + fn + m^2 \pm \sqrt{(3mn - fm + fn + m^2)^2 - 4(m + n)(fmn + 2m^2n - fm^2)}}{2(m + n)}
\]

\[
= \frac{3mn - fm + fn + m^2 \pm \sqrt{(m - n)^2(m + f)^2}}{2(m + n)}.
\]

Equation (3) is a quadratic equation with complex coefficients.

It follows that if expression \((m - n)^2(m + f)^2\) does not equal 0, then (3) will have two solutions. In the present case it necessarily holds that \(m \neq -f\), and hence it follows that points \(F\) and \(M\) are not the ends of the diameter of circle \(\omega\). This means that the center, \(O\), of the circle does not belong to straight line \(MF\) (the line \(BF\)). In this case the two solutions of the equation are:

\[
z_1 = \frac{3mn - fm + fn + m^2 + (m - n)(m + f)}{2(m + n)} = \frac{2mn + 2m^2}{2(m + n)} = m,
\]

and

\[
z_2 = \frac{3mn - fm + fn + m^2 - (m - n)(m + f)}{2(m + n)} = \frac{2mn - fm + fn}{m + n}.
\]

It is clear that the first solution is the complex coordinate of point \(M\), and the second solution is the coordinate of another point that belongs to straight line \(BC\) (denoted by \(V\)).
In other words: \( v = \frac{2mn - fm + fn}{m + n} \).

Similarly, one can prove that in the case where the center, \( O \), of circle \( \omega \) does not lie on straight line \( AF \), circle \( \sigma_{PQ} \) will intersect straight line \( AD \) at two points: 1) at point \( N \), and 2) at some other point (designated as \( W \)) whose complex coordinate can be expressed as \( w = \frac{2mn - fn + fm}{m + n} \).

If \((m - n)^2 (m + f)^2 = 0\) holds, then Equation (3) has a single solution. For two different points \( M \) and \( N \) located on unit circle \( \omega \), there holds \( m \neq n \), therefore necessarily there holds \( m = -f \). In other words, points \( F \) and \( M \) are the ends of a diameter of circle \( \omega \), and therefore center \( O \) of the circle belongs to straight line \( BF \).

In this case, the only solution of the equation is:

\[
\frac{3mn - fm + fn + m^2}{2(m + n)} = \frac{3mn + m^2 - mn + m^2}{2(m + n)} = \frac{2mn + 2m^2}{2(m + n)} = m.
\]

In other words, in this case, line \( BF \) is tangent to circle \( \sigma_{PQ} \) at point \( N \).

(b) Let us prove that points \( V \), \( T \), and \( W \) lie on the same straight line (see Figure 7).

We shall use the following formula, which gives the relation between the coordinates of any three collinear points \( A(a) \), \( B(b) \), and \( C(c) \):

\[
a \overline{(b - \overline{c})} + b \overline{(c - \overline{a})} + c \overline{(a - b)} = 0. \tag{IV}
\]

According to this formula, points \( V \), \( T \), and \( W \) are collinear provided the following equality holds:

\[
v \overline{(\overline{t} - \overline{v})} + t \overline{(\overline{w} - \overline{t})} + w \overline{(\overline{v} - \overline{t})} = 0. \tag{4}
\]

Let us first calculate the conjugates of coordinates \( v \) and \( w \):

\[
\overline{v} = \frac{2mn - fm + fn}{m + n} = \frac{2}{m + n} \left( \frac{2}{m + n} - \frac{1}{m} + \frac{1}{n} \right) = \frac{2f - n + m}{f (m + n)}
\]

and similarly

\[
\overline{w} = \frac{2f - m + n}{f (m + n)}.
\]

We substitute the expressions for \( t, \overline{t}, v, \overline{v}, w, \) and \( \overline{w} \) into (4), to obtain:

\[
\frac{2mn - fm + fn}{m + n} \left( \frac{2}{m + n} - \frac{2f - m + n}{f (m + n)} \right) + \frac{2mn}{m + n} \left( \frac{2f - m + n}{f (m + n)} - \frac{2f - n + m}{f (m + n)} \right) + \frac{2mn - fn + fm}{m + n} \left( \frac{2f - n + m}{f (m + n)} - \frac{2}{m + n} \right) = 0.
\]
After simplifying the left-hand side, we have:

\[
\begin{aligned}
&\frac{2mn - fm + fn}{m + n} \cdot \frac{m - n}{f(m + n)} + \frac{2mn}{m + n} \cdot \frac{2n - 2m}{f(m + n)} \\
+\frac{2mn - fn + fm}{m + n} \cdot \frac{m - n}{f(m + n)} = 0, \\
\frac{m - n}{f(m + n)} \cdot \frac{2mn - fm + fn - 4mn + 2mn - fn + fm}{=0} = 0.
\end{aligned}
\]

We have thus obtained \( 0 = 0 \).

In other words, (IV) is satisfied, and therefore points \( V, T, \) and \( W \) must be on the same straight line, and segment \( VW \) is a diameter of circle \( \sigma_{PQ} \).

(c) In (a), we proved that circles \( \omega \) and \( \sigma_{PQ} \) intersect at points \( M \) and \( N \), and therefore \( r_\omega = OM \) and \( r_{\sigma_{PQ}} = TM \). Let us find the distance, \( OT \), between the centers of circles \( \omega \) and \( \sigma_{PQ} \):

\[
OT^2 = (t - 0) (\bar{t} - \bar{0}) = \left( \frac{2mn}{m + n} - 0 \right) \left( \frac{2}{m + n} - 0 \right) = \frac{4mn}{(m + n)^2}.
\]

Now we calculate the sum \( r_\omega^2 + r_{\sigma_{PQ}}^2 \):

\[
r_\omega^2 + r_{\sigma_{PQ}}^2 = 1 - \left( \frac{m - n}{m + n} \right)^2 = \frac{(m + n)^2 - (m - n)^2}{(m + n)^2} = \frac{4mn}{(m + n)^2}.
\]

Therefore, the equation \( r_\omega^2 + r_{\sigma_{PQ}}^2 = OT^2 \) holds, and, specifically, \( OM^2 + TM^2 = OT^2 \) holds.

It thus follows that angle \( \angle OMT \) is a right angle, and therefore line \( OM \) is tangent to circle \( \sigma_{PQ} \), and line \( TM \) is tangent to \( \omega \).

We obtained that the tangents to circles \( \omega \) and \( \sigma_{PQ} \) at the point of their intersection, \( M \), are perpendicular to each other.

Therefore the circles are perpendicular to each other. \( \Box \)

Conclusions from Theorem 2.

(1) We obtained that the two segments \( PQ \) and \( VW \) are diameters of circle \( \sigma_{PQ} \). Therefore their lengths are equal, and they bisect each other (at point \( T \)). It follows that quadrilateral \( PVQW \) is a rectangle (see Figure 8).

Note: Rectangle \( PVQW \) (in which two opposite vertices are Pascal points) is usually different from the rectangle inscribed in quadrilateral \( ABCD \) in such a manner that its sides are parallel to diagonals \( AC \) and \( BD \), which are perpendicular to each other. (In Figure 8 rectangle \( PXYZ \) is inscribed in the quadrilateral and its sides are parallel to the diagonals of the quadrilateral.)

(2) For any quadrilateral, \( ABCD \), with perpendicular diagonals and any circle, \( \omega_i \), that forms a pair of Pascal points \( P_i \) and \( Q_i \) on sides \( AB \) and \( CD \), one can define a rectangle that is inscribed in quadrilateral \( ABCD \) as follows:

We construct a Pascal points circle \( \sigma_{P_i Q_i} \) that intersects sides \( BC \) and \( AD \) at points \( V_i \) and \( W_i \) (in addition to points \( N_i \) and \( M_i \)). Points \( P_i, V_i, Q_i, \) and \( W_i \) define a rectangle inscribed in quadrilateral \( ABCD \).
(3) Let $\omega$ be a circle that passes through points $E$ and $F$, intersects sides $BC$ and $AD$ at points $M$ and $N$, respectively, and forms Pascal points $P$ and $Q$ on sides $AB$ and $CD$. $T$ is the point of intersection of the tangents to circle $\omega$ at points $M$ and $N$. In this case, the circle whose center is at point $T$ and whose radius is segment $TM$ is the Pascal points circle $\sigma_{PQ}$.

Explanation: In Theorem 1 we proved that the tangents to circle $\omega$ at points $M$ and $N$ intersect in the middle of segment $PQ$ (at point $T$). In Theorem 2 we proved that Pascal points circle $\sigma_{PQ}$ passes through points $M$ and $N$.

**Theorem 3.**

Let $ABCD$ be a quadrilateral with perpendicular diagonals in which $E$ is the point of intersection of the diagonals and $F$ is the point of intersection of the extensions of sides $BC$ and $AD$: $\omega_{EF}$ is a circle whose diameter is segment $EF$; Circle $\omega_{EF}$ intersects sides $BC$ and $AD$ at points $M_0$ and $N_0$, respectively, and forms Pascal points $P_0$ and $Q_0$ on sides $AB$ and $CD$; $\sigma_{P_0Q_0}$ is the Pascal points circle of points $P_0$ and $Q_0$. Then:

(a) Circle $\sigma_{P_0Q_0}$ intersects the sides of quadrilateral $ABCD$ at 8 points, as follows: It intersects side $AB$ at points $P_0$ and $M_1$, side $BC$ at $M_0$ and $V_0$, side $CD$ at points $Q_0$ and $N_1$, side $AD$ at $N_0$ and $W_0$. (In Figure 9, one can observe the fours points of intersection mentioned in Theorem 2, $N_0$, $M_0$, $V_0$, and $W_0$, and also two additional points of intersection, $M_1$ and $N_1$).

(b) Chords $V_0N_0$, $W_0M_0$, $Q_0M_1$, and $P_0N_1$ of the circle intersect at point $E$. 

Figure 8.
Properties of a Pascal points circle in a quadrilateral with perpendicular diagonals

The center, $O$, of circle $\omega_{EF}$ does not belong to straight lines $FB$ and $FA$. Therefore, from Theorem 2, circle $\sigma_{P_0Q_0}$ intersects each of the sides $BC$ and $AD$ at two points (at points $M_0$ and $V_0$, and at points $N_0$ and $W_0$, respectively). Therefore, it remains to be proven that $\sigma_{P_0Q_0}$ intersects each of the other two sides at two points.

We will first prove one additional property that holds for the points of intersection of circle $\sigma_{P_0Q_0}$ with sides $BC$ and $AD$. We will show that chords $V_0N_0$ and $W_0M_0$ both pass through point $E$.

We choose a system of coordinates such that circle $\omega_{EF}$ is the unit circle ($O$ is the origin and the radius, $OE$, equals 1).

From formula (IV) in the proof of Theorem 2, points $N_0(n)$, $E(e)$ and $V_0(v)$ are collinear provided the following equality holds:

$$n(\overline{e} - \overline{v}) + e(\overline{v} - \overline{n}) + v(\overline{n} - \overline{e}) = 0.$$

For $e$, $\overline{e}$, $v$, and $\overline{v}$ there holds: $e = -f$ (because segment $EF$ is the diameter of the unit circle), $\overline{e} = -\frac{1}{f}$, $v = \frac{2mn - fm + fn}{m + n}$, and $\overline{v} = \frac{2f - n + m}{f(m + n)}$ (see the proof of Theorem 2).

We substitute these expressions in the left-hand side of the formula above, and
obtain:
\[
n (\overline{e} - \overline{v}) + e (\overline{n} - \overline{v}) + v (\overline{n} - \overline{e}) = n \left( -\frac{1}{f} - \frac{2f - n + m}{f(m + n)} \right) - f \left( \frac{2f - n + m}{f(m + n)} - \frac{1}{n} \right) + \frac{2mn - fm + fn}{m + n} \left( \frac{1}{n} + \frac{1}{f} \right)
\]
\[
= n \cdot \frac{-2m - 2f}{f(m + n)} - f \cdot \frac{fn - n^2 + mn - fm}{fn(m + n)} + \frac{(2mn - fm + fn)(f + n)}{fn(m + n)}
\]
\[
= \frac{0}{fn(m + n)} = 0.
\]

In other words, the equality holds and therefore the points \(N_0, E,\) and \(V_0\) are collinear (see Figure 9).

Similarly, we also prove that points \(M_0, E,\) and \(W_0\) are collinear.

To find the remaining two points of intersection, we follow the following path:

The first stage is to find two points that can be candidates for the intersection of circle \(\sigma_{P_0Q_0}\) with sides \(AB\) and \(CD\). The second stage is to prove that these two points are really the points of intersection of circle \(\sigma_{P_0Q_0}\) with sides \(AB\) and \(CD\).

It is reasonable to assume that the property satisfied for the four points of intersection of circle \(\sigma_{P_0Q_0}\) with sides \(BC\) and \(AD\) shall also hold for the four points of intersection of circle \(\sigma_{P_0Q_0}\) with sides \(AB\) and \(CD\). Therefore, at the first stage we shall choose points \(M_1\) and \(N_1\) to be our candidates, which are the intersection points of line \(AB\) with line \(Q_0E\) and line \(CD\) with line \(P_0E\), respectively.

At the second stage, we shall prove that the points \(M_1\) and \(N_1\) belong to circle \(\sigma_{P_0Q_0}\).

Using formula (IV) we obtain the equation of straight line \(AB\).

The formula holds for three collinear points \(A(a), B(b),\) and \(C(c)\). If we replace the coordinate of point \(C\) by the coordinate of some point \(Z(z)\) that belongs to straight line \(AB\), we obtain the equation of \(AB\):

\[
a (\overline{b} - \overline{z}) + b (\overline{z} - \overline{a}) + z (\overline{a} - \overline{b}) = 0.
\]

This can be put in the form:

\[
\overline{z} = \frac{a - \overline{b} - \overline{a}b}{a - b} z + \frac{a\overline{b} - \overline{a}b}{a - b}.
\]

Let us express the coordinates of \(A(a)\) and \(B(b)\) (and their conjugates) using the coordinates of points \(F, E, K, L, M,\) and \(N\), which lie on the unit circle. We shall use the formulas (I) from the proof of Theorem 2.

In our case, segments \(K_0L_0\) and \(EF\) are diameters of circle \(\omega_{EF}\). Therefore, there holds that \(k = -l\) and \(e = -f\). The following expressions are therefore obtained:

\[
a = \frac{2nl + fl - fn}{n + l}, \quad \overline{a} = \frac{2f + n - l}{f(n + l)},
\]
\[
b = \frac{2ml + fl + fm}{l - m}, \quad \overline{b} = \frac{2f + m + l}{f(m - l)}.
\]
We substitute these expressions into (V) to obtain:

\[ z = \frac{2f + n - l}{2nl + fl - fn} \cdot \frac{f (n + l)}{n + l} - \frac{2f + m + l}{2ml + fl + fm} \cdot \frac{f (m + l)}{m + l} \cdot \frac{z}{l - m} + \frac{2f + m + l}{2ml + fl + fm} \cdot \frac{f (n + l)}{n + l} - \frac{2f + n - l}{2nl + fl - fn} \cdot \frac{f (m - l)}{m - l} \cdot \frac{z}{l - m}. \]

After simplifying, we obtain:

\[ z = \frac{fm - fn - 2fl - ml - nl}{fl (fm + fn + 2mn + ml - nl)} \cdot \frac{z}{l - m} + \frac{2fml + 2f2l + 2mnfl - f2n + f2m + n2 - ml2}{fl (fm + fn + 2mn + ml - nl)} + \frac{2fml + 2f2l + 2mnfl - f2n + f2m + n2 - ml2}{fl (fm + fn + 2mn + ml - nl)} \cdot \frac{z}{l - m}. \]

Similarly, we can obtain the equation of line \( QE \): we replace the letters \( a \) and \( b \) in (V) with the letters \( e \) and \( q \) to obtain:

\[ z = \frac{q - \sigma}{q - e} + \frac{q \sigma - qe}{q - e}. \]

In our case there holds: \( e = -f \) and \( \sigma = -\frac{1}{f} \).

Point \( Q \) is the point of intersection of straight lines \( KM \) and \( LN \). In addition, in our case there holds that \( k = -l \). Therefore, from the formulas (I) for \( q \) and \( \sigma \), we obtain the following expressions:

\[ q = \frac{2mn + ml - nl}{m + n} \quad \text{and} \quad \sigma = \frac{2l + n - m}{l (m + n)}. \]

We substitute these expressions in the equation of straight line \( QE \), and obtain:

\[ z = \frac{2l + n - m}{l (m + n)} + \frac{1}{f} \cdot \frac{2mn + ml - nl}{m + n} \cdot \frac{1}{l (m + n)} \cdot \frac{z}{l - m} + \frac{2fml + 2f2l + 2mnfl - f2n + f2m + n2 - ml2}{fl (2mn + ml - nl + fm + fn)} + \frac{2fml + 2f2l + 2mnfl - f2n + f2m + n2 - ml2}{fl (2mn + ml - nl + fm + fn)} \cdot \frac{z}{l - m}. \]

and after simplifying:

\[ z = \frac{2f + fn - fm + ml + nl}{fl (2mn + ml - nl + fm + fn)} \cdot \frac{z}{l - m} + \frac{2f2l + f2n - f2m - 2mnfl - ml2 + n2}{fl (2mn + ml - nl + fm + fn)} + \frac{2f2l + f2n - f2m - 2mnfl - ml2 + n2}{fl (2mn + ml - nl + fm + fn)} \cdot \frac{z}{l - m}. \]

By equating the right-hand sides of (5) and (6), we obtain an expression for complex coordinate \( z_{M_1} \) of the intersection point of straight lines \( AB \) and \( QE \):

\[ z_{M_1} = \frac{f2n - f2m - 2mn - fml - fnl}{fm - fn - 2fl - ml - nl} \cdot \frac{z}{l - m} + \frac{f2n - f2m - 2mn - fml - fnl}{fm - fn - 2fl - ml - nl} \cdot \frac{z}{l - m} + \frac{f2n - f2m - 2mn - fml - fnl}{fm - fn - 2fl - ml - nl} \cdot \frac{z}{l - m}. \]

The expression for the conjugate of \( m_1 \) is:

\[ \overline{m}_1 = \frac{f (nl - ml - 2f2 - fml - fn)}{f (nl - ml - 2f2 - fml - fn)}. \]

We now prove that point \( M_1 \) belongs to circle \( \sigma_{P_0Q_0} \).

From formula (1) in the proof of Theorem 2, the equation of circle \( \sigma_{P_0Q_0} \) is:
In summary, we have shown that the four chords $\gamma$, $\delta$, $\epsilon$, and $\eta$ of circle $\sigma$ are similar terms, we obtain:

\[
\frac{z - \frac{2mn}{m+n}}{z - \frac{2}{m+n}} = -\left(\frac{m-n}{m+n}\right)^2.
\]

Let us substitute the expressions for $m_1$ and $\overline{m}_1$ in the equation of the circle. We obtain:

\[
\left(\frac{f^2n - f^2m - 2lmn - fml - fnl}{f m - fn - 2fl - ml - nl} - \frac{2mn}{m+n}\right)
\times \left(\frac{ml - nl - 2f^2 - fm - fn}{f (nl - ml - 2mn - fm - fn) - \frac{2}{m+n}}\right)
= -\left(\frac{m-n}{m+n}\right)^2.
\]

Let us check if this equality is a true statement.

Observe the left-hand side of the equality. After adding fractions and collecting similar terms, we get:

\[
\frac{fn^2 - fm^2 - lm^2 - ln^2 + 2mn - 2m^2n - 2mn^2}{(fm - fn - 2fl - ml - nl) (m+n)}
\times \frac{ln^2 - fn^2 + 2fmn - 2fnl + 2fml}{(nl - ml - 2mn - fm - fn) (m+n)}.
\]

After factoring the expressions in the numerators, we obtain:

\[
\frac{(n-m) (fm + fn - nl + ml + 2mn)}{(fm - fn - 2fl - ml - nl) (m+n)} \cdot \frac{(m-n) (ml + nl - fm + fn + 2fl)}{(ml - nl - 2mn - fm - fn) (m+n)}
= -\left(\frac{m-n}{m+n}\right)^2 (-fm - fn + nl - ml - 2mn) (-ml - nl + fm - fn - 2fl)
\times \frac{(fm - fn - 2fl - ml - nl) (nl - ml - 2mn - fm - fn) (m+n)^2}{(fm - fn - 2fl - ml - nl) (nl - ml - 2mn - fm - fn) (m+n)^2}
= -\left(\frac{m-n}{m+n}\right)^2.
\]

We have obtained an identical expression on both sides of the equality. Therefore the last equality is a true statement, and therefore point $M_1$ belongs to circle $\sigma_{P_0Q_0}$. Since point $M_1$ belongs to line $AB$, it follows that circle $\sigma_{P_0Q_0}$ intersects straight line $AB$ at point $M_1$.

Similarly, we can prove that circle $\sigma_{P_0Q_0}$ intersects straight line $CD$ at point $N_1$.

In summary, we have shown that the four chords $V_0N_0$, $W_0M_0$, $Q_0M_1$, and $P_0N_1$ of circle $\sigma_{P_0Q_0}$ pass through point $E$, which is, therefore, their point of intersection.

\[\square\]

Conclusions from Theorems 1-3.

In proving Theorems 1-3 we considered a quadrilateral whose two opposite sides, $BC$ and $AD$, are not parallel, and we did not require any additional conditions concerning the remaining opposite sides.

In the case that sides $AB$ and $CD$ also intersect (we denote the point of their intersection by $G$), there will be circles that pass through points $E$ and $G$ and form Pascal points on sides $BC$ and $AD$. In this case, for these circles Theorems similar to Theorems 1-3 shall hold (the proofs of these theorems are similar to the proofs...
of Theorems 1-3). According to these theorems, the circle whose diameter is segment $EG$ (we denote it by $\psi_{EG}$) satisfies the following properties:

- (a) The circle $\psi_{EG}$ forms Pascal points on sides $BC$ and $AD$ (for now we denote these points by $P$ and $Q$, respectively).
- (b) The circle $\psi_{EG}$ and the circle whose diameter is segment $\overline{PQ}$ are perpendicular to each other, and they intersect at the points at which circle $\psi_{EG}$ intersects sides $AB$ and $CD$ (for now we denote these points by $M$ and $N$, respectively).
- (c) The circle whose diameter is segment $\overline{PQ}$ intersects sides $AB$ and $CD$ at points $V$ and $W$ (in addition to points $M$ and $N$). The four points $P$, $V$, $Q$, and $W$ define a rectangle inscribed in quadrilateral $ABCD$.

**Theorem 4.**

Let $ABCD$ be a quadrilateral with perpendicular diagonals in which $E$ is the point of intersection of the diagonals, $F$ is the point of intersection of the extensions of sides $BC$ and $AD$, and $G$ is the point of intersection of the extensions of the sides $AB$ and $CD$; $\omega_{EF}$ is a circle whose diameter is segment $EF$ which forms Pascal points $P_0$ and $Q_0$ on sides $AB$ and $CD$, respectively; $\sigma_{P_0Q_0}$ is the Pascal points circle of points $P_0$ and $Q_0$, which intersects the sides of quadrilateral $ABCD$ at the following eight points: $P_0$, $Q_0$, $M_0$, $N_0$, $V_0$, $W_0$, $M_1$, and $N_1$ (see Theorem 3); $\psi_{EG}$ is the circle whose diameter is segment $EG$. Then:

- (a) The circle $\psi_{EG}$ intersects sides $AB$ and $CD$ at points $M_1$ and $N_1$, respectively.
- (b) Circles $\psi_{EG}$ and $\sigma_{P_0Q_0}$ are perpendicular to each other.
- (c) Points $V_0$ and $W_0$ are the Pascal points formed by circle $\psi_{EG}$ on sides $BC$ and $AD$, respectively.
- (d) The angle between diameters $EF$ and $EG$ of circles $\omega_{EF}$ and $\psi_{EG}$ is equal to one of the two angles between diameters $P_0Q_0$ and $V_0W_0$ of circle $\sigma_{P_0Q_0}$ (in Figure 10, there holds: $\angle FEG = \angle V_0E_0Q_0$).

**Proof.**

(a) In circle $\sigma_{P_0Q_0}$, inscribed angle $\angle P_0M_1Q_0$ rests on diameter $P_0Q_0$. It therefore holds that $\angle P_0M_1Q_0 = 90^\circ$, and therefore also $\angle EM_1G = 90^\circ$. Hence it follows that point $M_1$ belongs to the circle whose diameter is $EG$ (circle $\psi_{EG}$). Similarly, $\angle P_0N_1Q_0 = 90^\circ$. Therefore $\angle EN_1G = 90^\circ$ and therefore $N_1 \in \psi_{EG}$.

(b) Inscribed angles $\angle P_0N_1M_1$ and $\angle P_0Q_0M_1$ rest on the same arc, $\overline{P_0M_1}$, in circle $\sigma_{P_0Q_0}$ (see Figure 11). Therefore $\angle P_0Q_0M_1 = \angle P_0N_1M_1$. In addition, for angle $\angle TQ_0M_1$ (which is another name for the angle $P_0Q_0M_1$) there holds: $\angle TQ_0M_1 = \angle TM_1Q_0$ (because they are the base angles of isosceles triangle $TQ_0M_1$). Therefore:

$$\angle TM_1Q_0 = \angle P_0N_1M_1.$$  \hfill (7)

Similarly, in circle $\psi_{EG}$ there holds that $\angle EN_1M_1 = \angle EGM_1$, and also $\angle O_1GM_1 = \angle O_1M_1G$. Therefore:

$$\angle O_1M_1G = \angle EN_1M_1.$$  \hfill (8)
Since angles $\angle P_0N_1M_1$ and $\angle EN_1M_1$, which appear in the right-hand side of equalities (7) and (8), are the same angle, therefore $\angle TM_1Q_0 = \angle O_1M_1G$.

Now, consider angle $\angle TM_1O_1$:

$$\angle TM_1O_1 = \angle TM_1Q_0 + \angle Q_0M_1O_1$$
$$= \angle TM_1Q_0 + (\angle Q_0M_1G - \angle O_1M_1G)$$
$$= \angle TM_1Q_0 + 90^\circ - \angle O_1M_1G$$
$$= 90^\circ.$$

We obtained that $\angle TM_1O_1 = 90^\circ$, and therefore $M_1T$ is tangent to circle $\psi_{EG}$, and $M_1O_1$ is tangent to circle $\sigma_{P_0Q_0}$. Hence it follows that circles $\sigma_{P_0Q_0}$ and $\psi_{EG}$ are perpendicular to each other.

(c) Circles $\sigma_{P_0Q_0}$ and $\psi_{EG}$ intersect at an additional point: $N_1$. Therefore the tangent to circle $\psi_{EG}$ at point $N_1$ also passes through the center, $T$, of circle $\sigma_{P_0Q_0}$. We obtained that the tangents to circle $\psi_{EG}$ at points $M_1$ and $N_1$ intersect at point $T$. Points $M_1$ and $N_1$ are the points of intersection of circle $\psi_{EG}$ with sides $AB$ and $CD$. Therefore, from Conclusion 3 of Theorem 2, we have that the circle whose center is point $T$ and whose radius is segment $TM_1$ is the Pascal points circle of the points formed by circle $\psi_{EG}$ on sides $AB$ and $CD$.

On the other hand, in Theorem 3, we have proven that Pascal points circle $\sigma_{P_0Q_0}$ passes through points $M_1$ and $N_1$, and its center is at point $T$.

Therefore the Pascal points circle of the points formed by circle $\psi_{EG}$ is circle $\sigma_{P_0Q_0}$.
In Theorem 3 we saw that circle $\sigma_{P_0Q_0}$ intersects side $BC$ at points $M_0$ and $V_0$ and also intersects side $AD$ at points $N_0$ and $W_0$.

Of the four chords that connect a point on side $BC$ with a point on side $AD$ ($M_0W_0$, $N_0V_0$, $M_0N_0$, and $V_0W_0$), only $V_0W_0$ passes through $T$, the center of the circle $\sigma_{P_0Q_0}$. In other words, only $V_0W_0$ is a diameter of circle $\sigma_{P_0Q_0}$. Therefore points $V_0$ and $W_0$ are Pascal points formed by circle $\psi_{EG}$ on sides $BC$ and $AD$.

(d) Let us prove that straight line $FE$ is perpendicular to diameter $V_0W_0$.

From Theorem 3, we have that segments $W_0M_0$ and $V_0N_0$ pass through point $E$. In circle $\sigma_{P_0Q_0}$, angles $\angle W_0M_0V_0$ and $\angle V_0N_0W_0$ are inscribed angles resting on diameter $V_0W_0$ (see Figure 10). Therefore, they are right angles.

We obtained that segments $W_0M_0$ and $V_0N_0$ in triangle $FV_0W_0$ are altitudes to sides $FV_0$ and $FW_0$, respectively, and that $E$ is their point of intersection. It follows that straight line $FE$ contains the third altitude (the altitude to side $W_0V_0$) of triangle $FV_0W_0$, and therefore $EF \perp V_0W_0$.

Similarly, one can prove that $EG \perp P_0Q_0$.

In summary, segments $EF$ and $EG$ are perpendicular to diameters $V_0W_0$ and $P_0Q_0$, respectively, of circle $\sigma_{P_0Q_0}$, and therefore angle $\angle FEG$ is equal to one of the angles between diameters $V_0W_0$ and $P_0Q_0$. □

Conclusion from Theorems 2-4.

In a quadrilateral, $ABCD$, in which diagonals are perpendicular and intersect at
point $E$, and the extensions of the opposite sides intersect at points $F$ and $G$, there holds: the Pascal points formed by circles $\omega_{EF}$ and $\psi_{EG}$ are the vertices of a rectangle inscribed in the quadrilateral (see Figure 12).

References


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