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The Relativity of Conics and Circles

Stefan Liebscher and Dierck-E. Liebscher

Abstract. Foci are defined for a *pair* of conics. They are the *six* vertices of the quadrilateral of common tangents. To be circle is a derived property of a pair of conics, too.

1. Introduction

The beloved properties of conics, which we start to discover at school, manifest themselves as projective *relations between two conics,* as soon as we use the embedding in the non-Euclidean Cayley-Klein geometries. Non-Euclidean geometry enables us to find real constructions of objects which otherwise lie in infinity or are imaginary.

Our school-level understanding of geometry is a play with ruler and compass on the Euclidean plane. Conics constitute the last topic to be tackled in this manner and the first that leads beyond its limits. The most prominent property of conics is the existence of foci.

In non-Euclidean geometry, the scale on the drawing plane is not constant. So, how can we define motions and determine congruency? We start by generating the group of motions by defining reflections [1]. Reflections define perpendiculars, and in Cayley-Klein geometries, a conic provides the reflections simply by requiring its own invariance [1]. Reflection of tangents to this absolute conic yield other tangents again, and the reflection of a point is the intersection of the reflection of its two tangents to the absolute conic. Two lines are perpendicular if one is reflected by the other onto itself. Such lines contain the others pole. The pole becomes the common intersection of all perpendiculars of its polar.

Geometries can now be classified by their absolute conics. In Euclidean geometry, this conic is a double line at infinity with complex fixed points, so it is not prominent. The line at infinity contains the poles of all lines and is the polar of all points. In Minkowski geometry, the fixed points are real, and become prominent in their role as directions of the world lines of light signals [2]. In Galilei geometry, these two fixed points coincide [5]. The Beltrami-Klein model of the hyperbolic plane uses a regular conic. To adapt to our Euclidean habits, it has the form of a circle. As we shall see, a conic acquires the properties of a circle when declared absolute.

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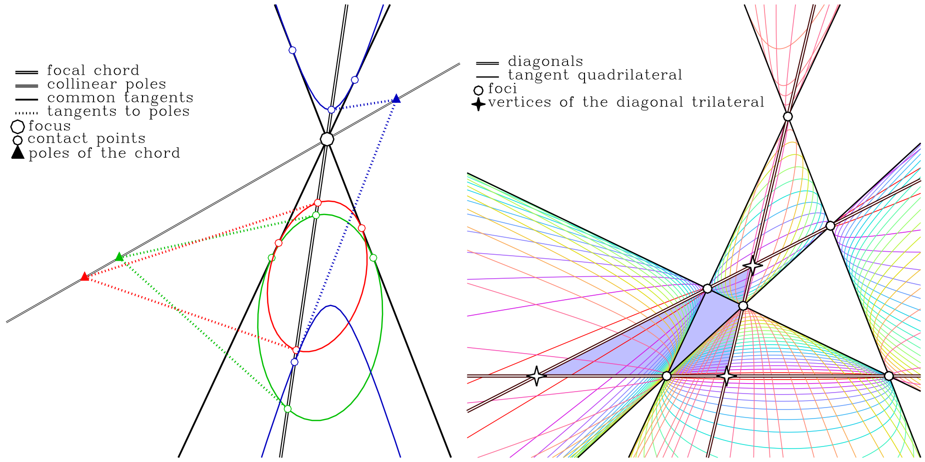


Left: Given a set of conics with two common tangents. The poles of any line passing the vertex of the tangents are concurrent with this vertex. Right: Two conics define four common tangents and generate a pencil of conics. Any two conics of the pencil have the same quadrilateral of common tangents with the same six vertices and three diagonals (dashed lines). The vertices are foci by this pole-collinearity property. If any conic of the pencil is declared absolute, the foci obtain the familiar metric properties of a focus, too.

Figure 1. Confocal conics for real common tangents

2. Foci as properties of pairs of conics

The focus of a conic is characterized by a particular property of its pencil of rays: The pole (with respect to the conic considered) of any line through the focus lies on the perpendicular to the line in the focus. In other words, the poles of a focal ray with respect to the two conics (the considered and the absolute) are collinear with the focus. This statement does not refer to the task of the absolute conic to define perpendicularity. It solely uses the pole-polar relation of two conics.

Foci thus become properties of pairs of conics instead of a single conic and the metric. Let us consider a tangent from the focus to one of the conics, so that its pole is the contact point. The pole by the other conic can be collinear with this contact point and the focus only when it lies on the tangent, too. That is, the line is tangent to both conics. A focus must be an intersection of tangents common to the conics, whether real or not. Consequently, the intersections of the common tangents are the foci (Fig. 1, left).

The foci of a pair of conics are the vertices of the quadrilateral of common tangents. A pair of conics has six foci, more precisely three pairs of opposite foci. Fig. 1, right, shows a pencil of confocal conics when all four common tangents are real lines. The diagonal lines of the tangent quadrilateral form a self-polar triangle: Its vertices and edges are pole-polar pairs (to *all* conics of the pencil). Its vertices are also the diagonal points of the quadrangle of intersections of any two conics of the pencil. Thus, the diagonal triangle is self-dual, too [3].

The consideration of metric properties of foci requires the promotion of one conic of the pencil as absolute, i.e. as generating the metric. For any other conic of the pencil and any focus, we obtain the three familiar properties:
(1) All poles of a line through a focus are collinear with the focus. The line connecting the focus with the poles of the reference line is perpendicular to the latter, independent of which conic of the pencil is taken as absolute.
(2) The lines through a focus are reflected in the tangents of any conic of the pencil onto lines though the opposite focus, independent of which conic of the pencil is taken as absolute.
(3) Yet more, the focus itself is reflected in the tangents of a chosen conic of the pencil onto the points of a circle around the opposite focus, independent of which conic of the pencil is taken as absolute. (This adds the gardener's rule to outline an ellipse using a rope attached to two pegs in the foci of the ellipse.)

3. Being circle as a relation between conics

Circles are usually understood by metric considerations, in particular the constant distance from some center. In non-Euclidean geometry, we refer to simpler projective properties. When symmetry is defined by an absolute conic, a circle should be symmetric with respect to all reflections on the lines through some center, i.e. the diameters. The tangents in the intersection of a diameter with the circle should be perpendicular to the diameter, i.e. the pole of a diameter with respect to the circle coincides with its pole relative to the absolute conic.

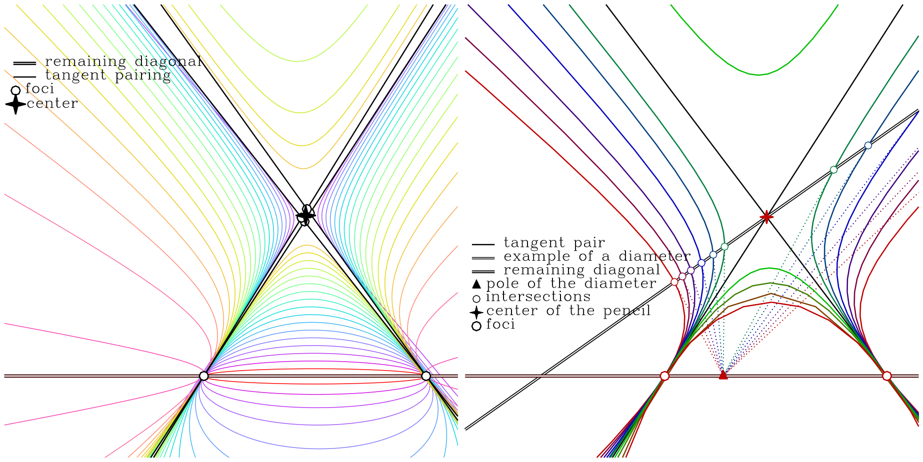
Again, we argue with the diameters tangent to the circle. The pole of such a diameter with respect to the circle is the point of contact.

When the poles with respect to the circle and to the absolute conic have to coincide, the point of contact with the circle is equally the point of contact with the absolute conic. A conic is a circle if it touches the absolute conic twice. The center of the circle is the intersection of the two tangents in the points of contact.

A set of concentric circles has a common center and a common pair of points of contact. This is the reason why any of the circles of the pencil can play the role of the absolute conic without any change in the pencil. The defining structure is a pair of tangents with a pair of contact points (Fig. 2). It is a quadrilateral formed by the two tangents and the double line connecting the contact points. Any two conics that show such a structure are circle with respect to each other. Two conics become circles to each other if two pairs of their foci collide and thus become the (common) midpoint of the circles. The third pair becomes the pair of contact points. In short: Two conics are circles to each other if they touch twice.

4. Confocal conics

This is an example to demonstrate the scope of projective phrasing. In Euclidean formulation, it is observed for two confocal ellipses that the tangents of the inner one intersect the outer ellipse in two points. When the tangent is reflected at the intersection point by the outer ellipse, it becomes the second tangent to the inner ellipse from this point. The projective phrasing shows without toil that a hyperbola



Left: When two pairs of foci coincide, the other pair becomes a pair of contact points common to all conics of the pencil. Its connection is the polar of the center (formed by the collision of the two pairs) with respect to all conics of the pencil. Right: The poles of a line passing the center coincide. We found the pencil of concentric circles.

Figure 2. Concentric circles for real common tangents and contact points

confocal to the ellipse yields the same, and that the unreflected absolute conic of the Euclidean plane can be replaced by any other conic confocal with the two just considered.

Given two conics, $\mathcal{K}$ and $\mathcal{L}$, they define their foci as vertices of the quadrilateral of common tangents. A third conic $\mathcal{C}$ is confocal to this pair, if it belongs to the pencil $\kappa\mathcal{K} + \lambda\mathcal{L}$, i.e. if it is tangent to the common quadrilateral.

Proposition: Given the conic $\mathcal{C}$ which determines perpendicularity. Any tangent t_1 to $\mathcal{K}$ is reflected at the intersection with a confocal with $\mathcal{K}$ and $\mathcal{C}$ conic $\mathcal{L}$ by the tangent to $\mathcal{L}$ into a tangent t_2 to $\mathcal{K}$. Proof: Any point $Q \notin \mathcal{K}$ determines two tangents $t_{1,2}$ to $\mathcal{K}$, whether real or not. Some conic $\mathcal{C}$ is chosen to determine perpendicularity. If the numbers $\alpha_{1,2} = \langle t_{1,2}, \mathcal{C}t_{1,2}\rangle$ have the same sign, we obtain the angular bisectors of the tangents $t_{1,2}$ at Q as

$$w_\pm \propto \sqrt{\alpha_2}\, t_1 \pm \sqrt{\alpha_1}\, t_2$$

Both bisectors are tangents to conics confocal to the pair $\{\mathcal{K}, \mathcal{C}\}$, in particular

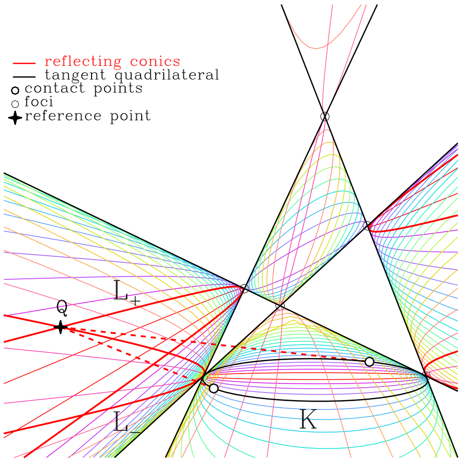
$$\mathcal{L}_\pm = \langle w_\pm, \mathcal{C}w_\pm\rangle\mathcal{K} - \langle w_\pm, \mathcal{K}w_\pm\rangle\mathcal{C}.$$

The contact points $\mathcal{L}_\pm w_\pm$ of the bisectors conicide with their intersection because expansion yields

$$\langle t_{1,2}, \mathcal{L}_\pm w_\pm\rangle = 0.$$

We obtained for the point Q two conics $\mathcal{L}_\pm$ which provide the required reflection of t_1 in t_2. $\square$

The two conics are determined by the quadrilateral of tangents of the pair $\{\mathcal{K}, \mathcal{C}\}$. In addition, because the two conics $\mathcal{L}_\pm$ confocal to $\{\mathcal{K}, \mathcal{C}\}$ do not change when



In a pencil of confocal conics, the tangents from any point Q to any conic $\mathcal{K}$ are reflected into each other by both conics $\mathcal{L}_\pm$ of the pencil that pass this point. At the intersection points of two conics $\mathcal{L}_\pm$ of the pencil, the two tangents to $\mathcal{L}_\pm$ are perpendicular if the absolute conic is anyone of the pencil, too.

Figure 3. Tangents of confocal conics

another conic of the pencil is chosen as absolute, the angular bisectors are independent of such a choice.

We conclude: In a pencil of confocal conics, a tangent to a conic $\mathcal{K}$ of the pencil from a point Q on any other conic of the pencil is reflected into the second tangent from Q to $\mathcal{K}$ at the tangents of the (two) conics of the pencil in Q, independent of which (third) conic of the pencil is promoted to serve as the metric-determining.

5. Summary

The definition of foci in non-Euclidean (precisely metric-projective) geometries reveals a structure basically independent of the particular explicit metric properties. The structure can be understood as a pure relation between conics.

(1) The quadrilateral of tangents common to two conics defines six points, which reveal the familiar properties of foci. The two poles of a line through a focus are collinear with the focus.

(2) The four lines of a quadrilateral define a pencil of conics touching the four lines. The intersection point of the four lines are the foci for any pair of conics of this pencil. For *any* line through a focus, the poles with respect to all the conics of the pencil are *collinear*.

(3) The diagonal lines of the quadrilateral form a self-polar triangle: Its vertices and edges are pole-polar pairs (to all conics of the pencil). Its vertices are also the diagonal points of the quadrangle of intersections of any two conics of the pencil. Thus, the diagonal triangle is self-dual, too.

(4) Independent of which conic of the pencil is taken as absolute, the three cited characteristics of foci are present.

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Stefan Liebscher: TNG Technology Consulting, Munich, Germany
E-mail address: `contact@stefan-liebscher.de`

Dierck-E. Liebscher: Leibniz-Institut für Astrophysik Potsdam, Germany
E-mail address: `deliebscher@aip.de`