A Remark on the Arbelos and the Regular Star Polygon

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Abstract. We give a condition that a regular star polygon \( \{ \frac{n}{2} \} \) can be constructed from an arbelos.

We consider to construct a regular star polygon \( \{ \frac{n}{2} \} \) from an arbelos. Let us consider an arbelos made by the three circles \( \alpha, \beta, \gamma \) with diameters \( AO, BO, AB \), respectively, for a point \( O \) on the segment \( AB \). Circles of radius \( ab/(a + b) \) are called Archimedean circles, where \( a \) and \( b \) are the radii of \( \alpha \) and \( \beta \), respectively. We call the perpendicular to \( AB \) at \( O \) the axis.

The circle touching \( \gamma \) internally, \( \alpha \) externally and the axis is Archimedean \( \delta \), which we describe by \( \varepsilon \) (see Figure 1). While the circle touching \( \beta \) internally and the tangents of \( \alpha \) from \( B \) is also Archimedean \( \varepsilon \), which is denoted by \( \varepsilon \). Therefore the figure made by \( \delta, \alpha \) and their tangents are congruent to the figure made by \( \varepsilon, \alpha \) and their tangents. We assume that \( C \) is the center of \( \alpha, D \) is the external center of similitude of \( \alpha \) and \( \delta \), and \( \theta = \angle BCD \). The congruence of the two figures implies that we can construct a regular star polygon \( \{ \frac{n}{2} \} \) with center \( C \) and adjacent vertices \( B \) and \( D \) if \( \theta = 2\pi/n \), while \( \cos \theta = a/(a + 2b) \). Therefore
we can construct a regular star polygon \( \{ \frac{n}{2} \} \) with center \( C \) and adjacent vertices \( B \) and \( D \) if and only if
\[
\frac{a}{a+2b} = \cos \frac{2\pi}{n}.
\]

Figure 2. \( \{ \frac{5}{2} \} \), \( b = \sqrt{5}a/2 \)

Figure 2 shows the case \( n = 5 \). The distance between \( D \) and the point of contact of \( \alpha \) and \( \delta \) equals \( BO \) by the congruence. A problem stating this fact can be found in [2].

References


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