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On the Extrema of Some Distance Ratios

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Abstract. In the plane, given a line ℓ and two distinct points A, B with B not on ℓ, we use inversion to study the extremal values of the ratio MA/MB when M varies on ℓ. The method also holds when an arbitrary curve $\mathcal{C}$ is substituted for ℓ, and we dwell upon the case when $\mathcal{C}$ is a circle.

1. Introduction

Fixing distinct points A, B not on a line ℓ, the construction of the points H, K of ℓ such that $HA/HB \leq MA/MB \leq KA/KB$ for all points M on ℓ has already received some attention. Two recent articles justify the same construction by using coordinates ([2]) or by a synthetic proof ([3]). Here we propose a totally different construction which readily leads to the exact values of HA/HB and KA/KB. In a second part, we extend the method, examining in particular the case when ℓ is replaced by a circle.

2. The construction

As in [2] and [3], we assume that AB and ℓ are not perpendicular; in contrast to B, the point A is allowed to be on ℓ. Let $\mathbf{I}$ denote the inversion in the circle with center B and radius BA. The line ℓ inverts into a circle Γ passing through B, and for any point M on ℓ, the point $M' = \mathbf{I}(M)$ is the point of intersection other than B of the line BM and Γ (see Figure 1).

As for distances, we have

$$MA = \frac{AB^2 \cdot M'A}{BM' \cdot BA} = AB \cdot M'A \cdot \frac{MB}{AB^2} = \frac{M'A \cdot MB}{AB}$$

(since $\mathbf{I}(A) = A$ and $BM \cdot BM' = BA^2$) and therefore

$$\frac{MA}{MB} = \frac{M'A}{AB}.$$

Thus, as M varies on ℓ, the ratio $\frac{MA}{MB}$ is extremal if and only if $M'A$ is and therefore $\frac{MA}{MB}$ attains its minimum (resp. maximum) at a unique point H (resp. K) of ℓ: if the diameter of Γ through A intersects Γ at H' and K' with $AH' < AK'$, the desired points H and K are $H = \mathbf{I}(H')$ and $K = \mathbf{I}(K')$.

In passing, note that the line $K'H'$ inverts into a circle $\mathbf{I}(K'H')$ through B, A, H, K; since $\ell = \mathbf{I}(\Gamma)$ and because Γ is cut orthogonally by its diameter $K'H'$, HK is a diameter of this circle $\mathbf{I}(K'H')$ passing though A and B. This result leads to the construction presented in [2] and [3].

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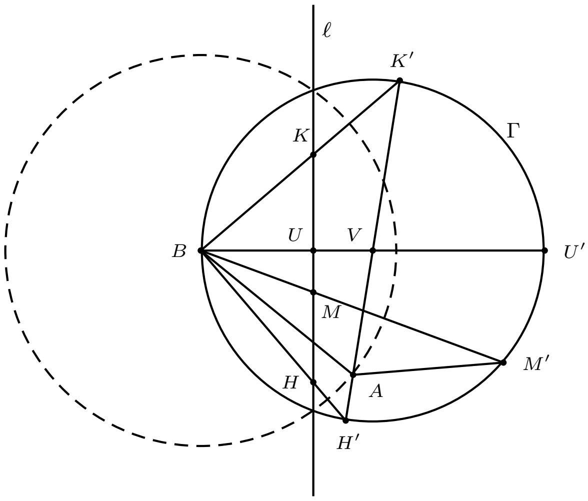

Figure 1.

3. The values of $\frac{HA}{HB}$ and $\frac{KA}{KB}$

We need the following preliminary result:

Lemma 1. *A is exterior or interior to Γ according as A and B are on the same side of ℓ or not.*

Proof. We introduce the orthogonal projection U of B onto ℓ and the midpoint V of the line segment BU' (where $U' = \mathbf{I}(U)$), that is, the center of Γ (Figure 1). With these notations, we have

$$4AV^2 + BU'^2 = 2(AB^2 + AU'^2), \qquad BU' = \frac{BA^2}{BU}, \qquad AU' = \frac{BA \cdot AU}{BU}$$

and a short calculation then gives

$$4AV^2 - 4BV^2 = \frac{2AB^2}{BU^2}(BU^2 + AU^2 - AB^2) = 4AB^2 \cdot \frac{AU}{BU}\cos(\angle AUB).$$

Thus, $AV < BV$ is equivalent to $\angle AUB > 90^\circ$, and the announced property follows. $\square$

For simplicity we set $\lambda = \frac{HA}{HB} = \min\{\frac{MA}{MB} : M \in \ell\}$ and $\mu = \frac{KA}{KB} = \max\{\frac{MA}{MB} : M \in \ell\}$ and now prove the following theorem:

Theorem 2. *If $d(X, \ell)$ denotes the perpendicular distance from the point X to the line ℓ, and $\varepsilon = +1$ if A, B are on the same side of ℓ, and $\varepsilon = -1$ if not, we have*

(i) $\lambda\mu = \frac{d(A,\ell)}{d(B,\ell)}$ *and* $\mu - \varepsilon\lambda = \frac{AB}{d(B,\ell)}$,

(ii) *the values of* λ *and* μ *are given by*

$$\lambda = \left| \frac{AB - \sqrt{AB^2 + 4\varepsilon d(A,\ell)d(B,\ell)}}{2d(B,\ell)} \right|,$$
$$\mu = \frac{AB + \sqrt{AB^2 + 4\varepsilon d(A,\ell)d(B,\ell)}}{2d(B,\ell)}.$$

Proof. Let r be the radius of Γ. First, suppose that A and B are on either side of ℓ (Figure 1). Then $AH' = \lambda AB$, $AK' = \mu AB$. Hence, A being interior to Γ, $(\lambda + \mu)AB = AH' + AK' = 2r$. Since $2r = BU' = \frac{BA^2}{BU} = \frac{AB^2}{d(B,\ell)}$, we obtain $\lambda + \mu = \frac{2r}{AB} = \frac{AB}{d(B,\ell)}$. Exchanging the roles of A and B, we see that $\lambda' = \min\{\frac{MB}{MA} : M \in \ell\} = \frac{1}{\mu}$ and $\mu' = \max\{\frac{MB}{MA} : M \in \ell\} = \frac{1}{\lambda}$ satisfy $\lambda' + \mu' = \frac{AB}{d(A,\ell)}$, that is, $\frac{\lambda+\mu}{\lambda\mu} = \frac{AB}{d(A,\ell)}$. The result (i) is readily deduced. Now, λ and μ are the roots of the quadratic equation $x^2 d(B,\ell) - xAB + d(A,\ell) = 0$ with $\lambda < \mu$ and (ii) follows.

Second, suppose that A and B are on the same side of ℓ (Figure 2). Then $AK' - AH' = 2r$ so that, as above, $\mu - \lambda = \frac{AB}{d(B,\ell)} = \lambda\mu\frac{AB}{d(A,\ell)}$. Thus (i) remains true. This time, $-\lambda$ and μ are the roots of $x^2 d(B,\ell) - xAB - d(A,\ell) = 0$ and (ii) holds.

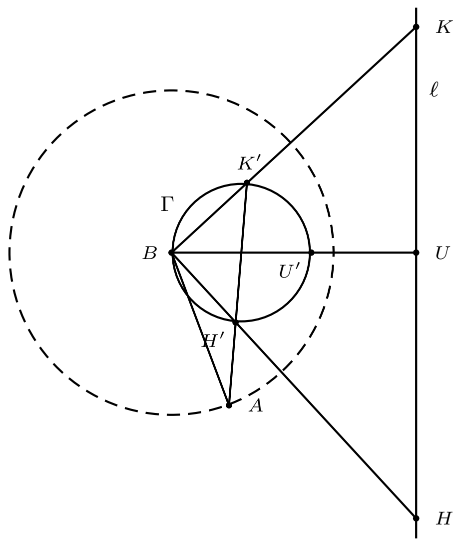

Figure 2

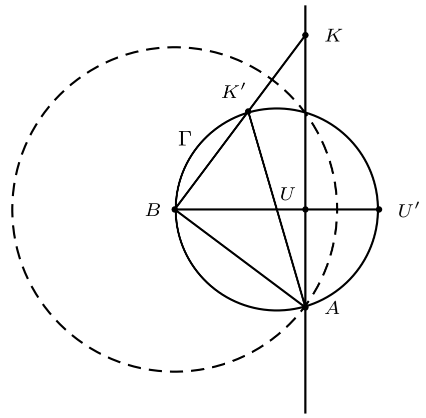

Figure 3

Lastly, if A is on ℓ, the formulas continue to hold: they give the expected values $\lambda = 0$ and $\mu = \frac{AB}{d(B,\ell)} = \frac{AB}{BU}$: indeed, because AK' is a diameter of Γ, we have $\angle ABK = 90°$, hence BU is the altitude from B in the right-angled ABK and so $\frac{KA}{KB} = \frac{AB}{BU}$ (Figure 3). $\square$

4. An extension and a consequence

We quickly examine the discarded case when AB is perpendicular to ℓ. Of course, if A and B are symmetrical in ℓ, then $\frac{MA}{MB} = 1$ for all points M of ℓ. Otherwise, supposing for example that A, B are on the same side of ℓ with $AU < BU$, then $\frac{MA}{MB}$ has no maximum when M varies on ℓ, but $\sup\left\{\frac{MA}{MB} : M \in \ell\right\} = 1$ while

$$\frac{MA^2}{MB^2} = \frac{AU^2 + UM^2}{BU^2 + UM^2} = \frac{UA^2}{UB^2} + \frac{MU^2(UB^2 - UA^2)}{BU^2(BU^2 + UM^2)} \geq \frac{UA^2}{UB^2}$$

so that $\min\left\{\frac{MA}{MB} : M \in \ell\right\} = \frac{UA}{UB}$. The other cases can be treated in a similar way. It is readily checked that if we replace max by sup and min by inf in the definition of λ and μ, the results (i) and (ii) of the theorem continue to hold.

Taking this extension of the theorem into account, we can deduce a nice corollary about conics:

Corollary 3. *Let $\mathcal{K}$ be the conic with focus F, associated directrix ℓ and eccentricity e. For P on $\mathcal{K}$, let $\lambda(P) = \inf\left\{\frac{MF}{MP} : M \in \ell\right\}$ and $\mu(P) = \sup\left\{\frac{MF}{MP} : M \in \ell\right\}$. Then we have*

$$\lambda(P) = |\mu(P) - e|. \tag{1}$$

Proof. Since $PF = ed(P, \ell)$, part (i) of the theorem gives $\mu(P) - \lambda(P) = e$ if $\mathcal{K}$ is a parabola or an ellipse ($e \leq 1$).

If $\mathcal{K}$ is a hyperbola, then $\mu(P) - \lambda(P) = e$ or $\mu(P) + \lambda(P) = e$ depending on the branch of $\mathcal{K}$ containing P, and so $\lambda(P) = |\mu(P) - e|$. $\square$

Note that relation (1) does not characterize the points of $\mathcal{K}$ ((1) remains valid for a point P of the reflection of $\mathcal{K}$ in ℓ).

5. Generalizing the method

In the plane, let $\mathcal{C}$ be any curve and A, B two fixed points with B not on $\mathcal{C}$. Let $\mathcal{C}' = \mathbf{I}(\mathcal{C})$ where $\mathbf{I}$ denotes the inversion in the circle with center B and radius BA. Mimicking the calculations of the second paragraph above, we readily obtain that for M on $\mathcal{C}$, we have $\frac{MA}{MB} = \frac{M'A}{AB}$ where $M' = \mathbf{I}(M)$. Thus, the ratio $\frac{MA}{MB}$ attains an extremum on $\mathcal{C}$ if and only if $M'A$ does on $\mathcal{C}'$.

This can be applied when $\mathcal{C}$ is a circle. In that case, the ratio $\frac{MA}{MB}$ is constant as M describes $\mathcal{C}$ if and only if A is the center of the circle $\mathbf{I}(\mathcal{C})$, that is, if and only if A, B are inverse in $\mathcal{C}$; otherwise the ratio $\frac{MA}{MB}$ attains its maximum (resp. minimum) at a unique point of the circle $\mathcal{C}$, which can be constructed with the help of the inverse of $\mathcal{C}$ in the same way as in the case of the line. The reader will fill in the details or find them in [1].

6. An alternative construction when $\mathcal{C}$ is a circle

It is worth mentioning a close but simpler construction that avoids the construction of $\mathbf{I}(\mathcal{C})$. We suppose that A, B are not inverse in the circle $\mathcal{C}$ and introduce the inversion $\mathbf{J}$ with center B leaving $\mathcal{C}$ globally invariant. Let $A' = \mathbf{J}(A)$. For any point M of $\mathcal{C}$, $M' = \mathbf{J}(M)$ is on $\mathcal{C}$ as well and

$$MA = \frac{|p| \cdot M'A'}{BM' \cdot BA'}$$

where p is the power of B with respect to $\mathcal{C}$ (Figures 4 and 5). Since $|p| = BM \cdot BM'$, we see that

$$\frac{MA}{MB} = \frac{A'M'}{A'B}. \tag{2}$$

If the diameter of $\mathcal{C}$ passing through A' intersects the circle in H' and K' with $A'H' < A'K'$, it follows that $H = \mathbf{J}(H')$ and $K = \mathbf{J}(K')$ satisfy

$$\frac{HA}{HB} = \min\left\{\frac{MA}{MB} :\ M \in \mathcal{C}\right\} \qquad \text{and} \qquad \frac{KA}{KB} = \max\left\{\frac{MA}{MB} :\ M \in \mathcal{C}\right\}.$$

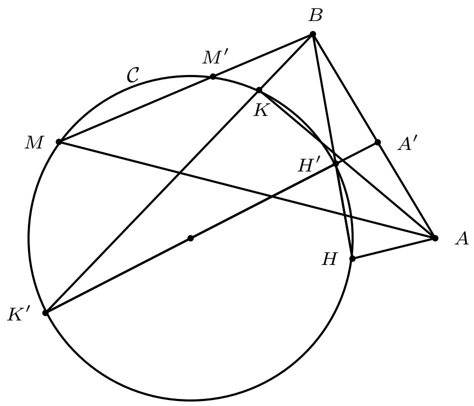

Figure 4

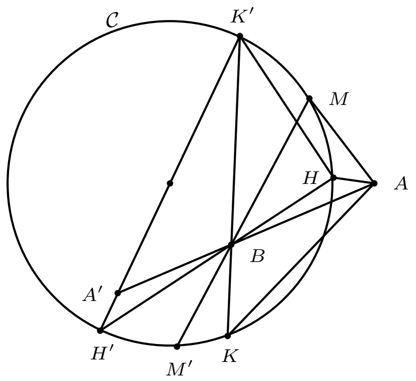

Figure 5

To conclude, we examine a nice application in the case when A is a fixed point on the circle $\mathcal{C}$ and B is not on $\mathcal{C}$ (Figure 6). Of course, we consider only the point K of $\mathcal{C}$ where $\frac{MA}{MB}$ attains its maximum. Let r be the radius of $\mathcal{C}$. From (2) we have

$$\frac{KA}{KB} = \frac{A'K'}{A'B} = \frac{2r}{A'B}.$$

Now, let k be a positive real number. Then, $\max\left\{\frac{MA}{MB} :\ M \in \mathcal{C}\right\} = k$ if and only if $A'B = \frac{2r}{k}$, that is, if and only if B belongs to the conchoid of $\mathcal{C}$ for pole A and constant $\frac{2r}{k}$ (a limaçon of Pascal). In the particular case $k = 1$, the limaçon is a cardioid (Figure 7) and we obtain the following, perhaps new, characterization of the cardioid:

Proposition 4. *Let $\mathcal{L}$ be the cardioid with cusp at A and fixed circle $\mathcal{C}$. Then a point B, not on $\mathcal{C}$, is on $\mathcal{L}$ if and only if* $\max\left\{\frac{MA}{MB} :\ M \in \mathcal{C}\right\} = 1$.

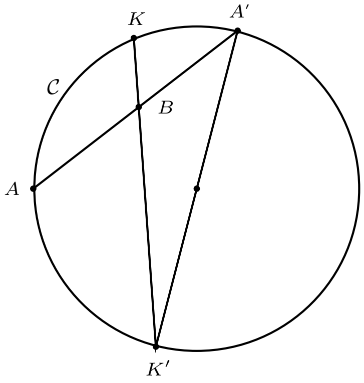

Figure 6

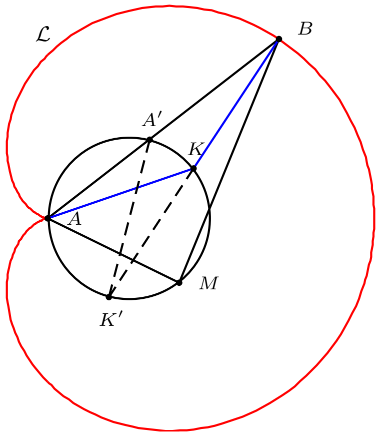

Figure 7

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