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Parabola Conjugate to Rectangular Hyperbola

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Abstract. We study the conjugation, naturally defined in a bitangent pencil of conics, and determine the triangles, which define corresponding pencils, for which the parabola member is conjugate to the rectangular hyperbola member of the pencil.

1. The bitangent pencil of conics

A *bitangent* family or pencil of conics is created by quadratic equations in the form of linear combinations

$$\alpha \cdot (f(x) \cdot g(x)) + \beta \cdot h(x)^2 = 0, \qquad \text{with } \alpha + \beta = 1, \tag{1}$$

where $\{f(x) = g(x) = h(x) = 0, x \in \mathbb{R}^2\}$ are equations of lines in general position and $\{\alpha, \beta\}$ are real constants ([1, II.p.194]). The family consists of conics,

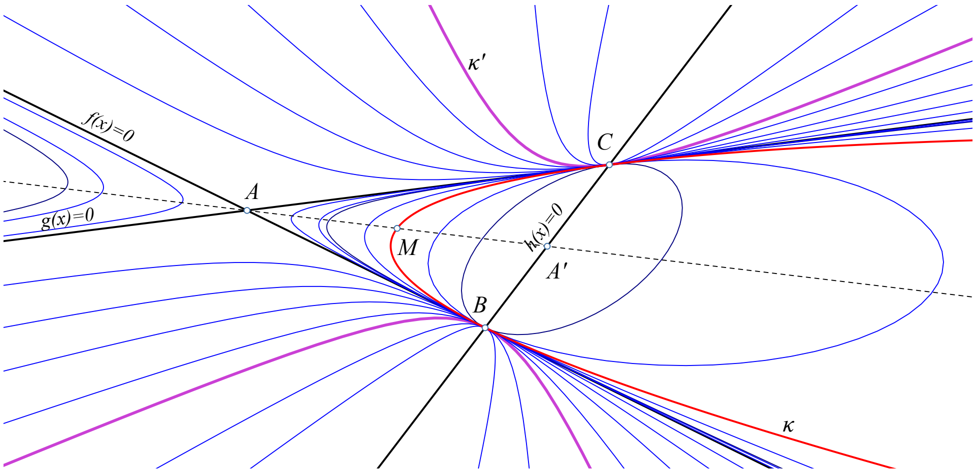

Figure 1. Bitangent family of conics

which are tangent to the lines $\{f = 0, g = 0\}$ at their intersections $\{B, C\}$ with the line $h = 0$. Thus, BC is a chord also common to all member-conics of the pencil, and the intersection point A of the two tangents, common to all members, is the pole of the line BC again with respect to all these members (see Figure 1). Since the median line AA' of the triangle ABC is the conjugate diameter of BC with respect to all members of the family, the centers of these conics lie on this line. Some general facts of a slightly more general kind of such families, in which the product of lines $f(x) \cdot g(x) = 0$ is replaced by a general conic $c(x) = 0$,

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are discussed in [7]. Every bitangent family has a unique parabola member, and a unique rectangular hyperbola member. The parabola κ passes through the middle M of the median AA' of the triangle ABC and the rectangular hyperbola κ' has its center at the projection of the orthocenter of the triangle ABC on this median line ([4], [6]). The aim of the article is to show that these two exceptional members of a bitangent pencil are *conjugate*, in a sense to be explained right below, only in the case the triangles ABC, formed by the three lines, are such that the median satisfies $BC = \sqrt{2} \cdot AA'$.

2. The perspectivities group of a triangle

A triangle ABC produces a set of three *harmonic perspectivities* (or *harmonic homologies* [8, I, p.223], [3, p.248]), which, together with the identity transformation, build a group of projective transformations having the vertices of the triangle for fixed points. The harmonic perspectivity f_B relative to the vertex B say, is

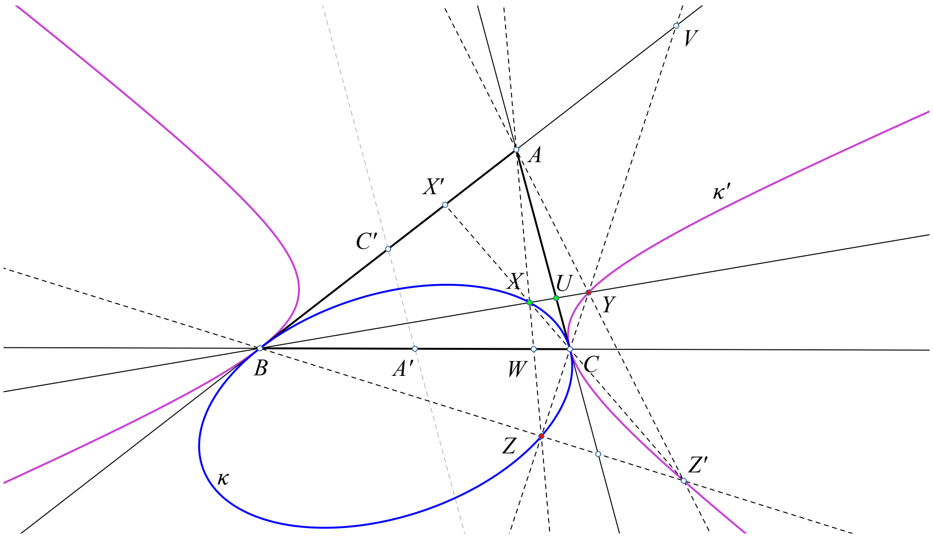

Figure 2. Harmonic perspectivity $Y = f_B(X)$ relative to B

defined by corresponding to a point X the point $Y = f_B(X)$ on line BX, so that the pairs $(B,U) \sim (X,Y)$ are harmonic, point U being the intersection of BX with AC (see Figure 2). The thus defined projective transformation f_B, leaves B and AC pointwise fixed and maps the line $A'C'$, of middles of sides $\{BC, BA\}$, to the line at infinity. Analogously are defined the maps $\{F_A, F_C\}$, and, from the properties of harmonic pencils ([5, p.88]), results the commutativity and the group property of these transformations (see Figure 2):

$$f_A^2 = 1, \quad f_A \circ f_B = f_B \circ f_A = f_C,$$

and the analogous properties for cyclic permutations of the letters $\{A, B, C\}$. Consider now the bitangent at $\{B, C\}$ family of conics $\mathcal{F}$ and the perspectivity $f_A(X) = Z$ and a member-conic κ. Since BC is the polar of point A with respect to κ, the map f_A leaves κ invariant. It is easy to see that the two other maps $\{f_B, f_C\}$ send κ to another member κ' of the family $\mathcal{F}$ (see Figure 2). In fact, if κ' is the family member through $Y = f_B(X)$ and $Z' = f_C(X)$, then points $\{X, Y, Z', Z = F_B(Z')\}$ define a complete quadrilateral ([5, p.100]), from whose harmonicity properties results that $Z' = f_A(Y)$. Consequently the same member κ' passes through $\{Y = f_B(X), Z' = f_C(X)\}$ and coincides with the image conic of κ under either of these maps: $\kappa' = f_B(\kappa) = f_C(\kappa)$. The last equality can be interpreted also by the conjugacy $f_C = h \circ f_B \circ h^{-1}$, where $h = h^{-1}$ is the affine reflection ([3, p.203]), defined by the lines $\{AA', BC\}$. Since these are conjugate diameters for each conic of the bitangent pencil $\mathcal{F}$, the transformation h leaves every such conic invariant.

We call two member-conics of the bitangent pencil $\mathcal{F}$, like $\{\kappa, \kappa'\}$, *conjugate*. Each of them results from the other by applying the perspectivity f_B or f_C. The property to be proved below is the following.

Theorem 1. *The parabola member of the family $\mathcal{F}$ is conjugate to the rectangular hyperbola member, if and only if, the triangle ABC and its median AA' satisfy $BC = \sqrt{2} \cdot AA'$.*

3. Hyperbola conjugate to ellipse

Using the notation and conventions adopted in the previous sections, we study the case in which the member-conic κ of the pencil $\mathcal{F}$ has a conjugate κ', which is a hyperbola. This is always the case for the parabola member of the pencil as well as for any other conic member κ contained in the same angular region, defined by the two lines $\{AB, AC\}$, in which lies the segment BC. The arguments used below are valid for all kinds of such conics and we can safely work assuming that κ is an ellipse, ultimately passing to the limiting case of the parabola. The following observations lead to the proof of the theorem.

(1) The directions of the asymptotes of the hyperbola κ' can be determined using the side-lines of the complementary triangle $A'B'C'$ of ABC (see Figure 3). In fact, if $\{B_1, B_2\}$ are the intersections of κ with line $A'C'$, send to infinity by f_B, then, because of the affine symmetry h, the points $\{C_1 = h(B_1), C_2 = h(B_2)\}$ are on the line $A'B'$, send to infinity by f_C. Using the commutativity properties of the maps $\{f_A, f_B, f_C, h\}$, we see immediately that $\{(BB_1, CC_2), (CC_1, BB_2)\}$ are pairs of lines parallel to asymptotic directions of the hyperbola κ'. In addition, it is seen that $\{(B_1, C_2, A), (C_1, B_2, A)\}$ are triples of collinear points, and that $\{B_1C_1, B_2C_2\}$ are parallel to the line BC, which implies that the intersections $I = (BB_2, CC_2)$ and $J = (BC_1, CB_1)$ are on line AA'.

(2) Consider now the intersections $\{E_1, E_2\}$ of κ' with line $B'C'$, send to infinity by f_A. We may assume that $\{AE_1, AE_2\}$ are parallel, respectively, to the asymptotic directions $\{BB_1, CC_1\}$ of κ'. For E_1 this means that

$$f_A(E_1) = f_B(B_1) \quad \Leftrightarrow \quad E_1 = f_A(f_B(B_1)) = f_C(B_1),$$

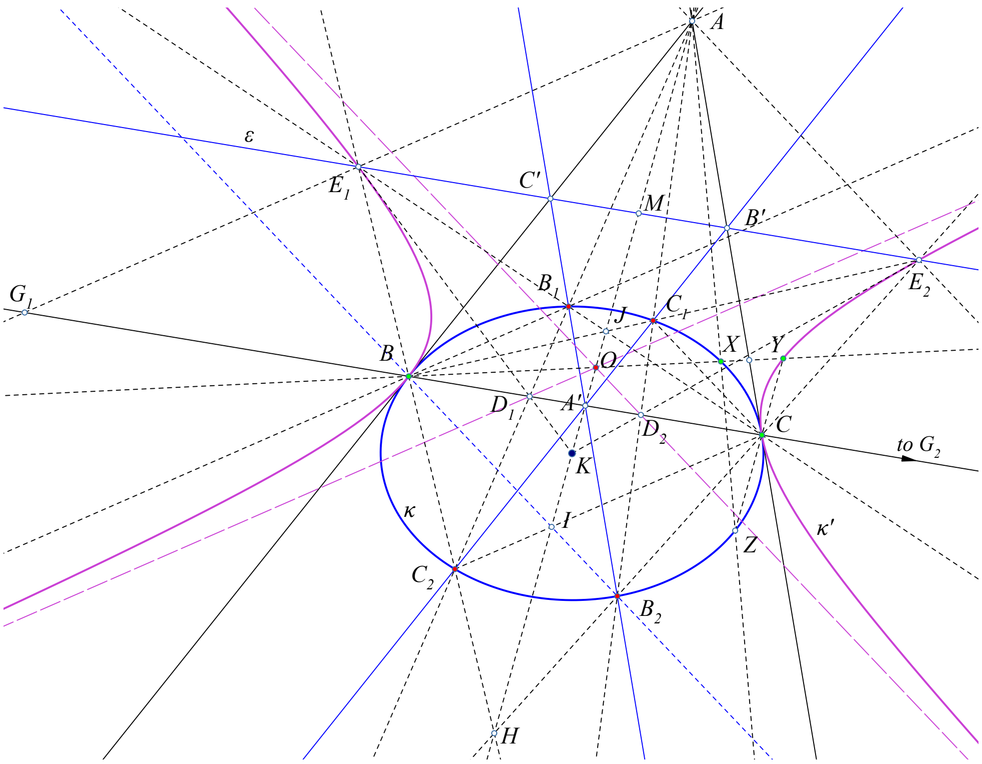

Figure 3. The conjugate to κ, hyperbola κ', and its asymptotes

so that $\{C, B_1, E_1\}$ are collinear points. Analogously is also seen that $\{B, C_1, E_2\}$ are collinear points.

(3) The polars of every point G on BC with respect to the conics $\{\kappa, \kappa'\}$ are the same, since they must pass through A and the harmonic conjugate G' of G with respect to (B, C). In particular, the intersection points $\{D_1 = (BC, AB_1), G_1 = (BC, AE_1)\}$ are harmonic conjugate with respect to (B, C) and the polar of one passes through the other i.e. AD_1 is the polar of G_1 and AG_1 is the polar of D_1 with respect to either conic. Analogous property holds also for the intersection points $\{D_2 = (BC, B_2C_2), G_1 = (BC, AE_2)\}$.

(4) Now, since the polar AG_1 of D_1 with respect to κ' is parallel to an asymptote, intersecting the hyperbola at E_1, the point D_1 must itself be on that asymptote and the tangent to the hyperbola κ' at E_1 must pass through D_1 ([2, p.281]). Hence if O is the center of the hyperbola, then OD_1 is the asymptote parallel to AG_1. Analogously, point $D_2 = (AB_1, BC)$ is on the tangent to κ' at E_2 and line OD_2 is the other asymptote.

(5) By the symmetry of the configuration with respect to the affine reflection h, we have that $\{E_2 = h(E_1), D_2 = h(D_1), G_2 = h(G_1)\}$ define segments $\{D_1D_2, E_1E_2, G_1G_2\}$ having their middles on AA'.

(6) By the parallelism of $\{AG_1, BB_1, OD_1, CC_2\}$ follows that these parallels intersect on E_1B a harmonic quadruple. Also, since the polar of D_1 passes through E_1, the polar of E_1 with respect to κ must pass through D_1, hence it coincides with the asymptote OD_1 of κ'. Analogously the polar with respect to κ of E_2 is the asymptote OD_2 of κ'. It follows then that O is the pole of $C'B'$ with respect to κ.

(7) Since the tangent E_1D_1 of κ' at E_1 passes through the intersection point D_1 of the diagonals of the trapezium BB_1CC_2 and the intersection point E_1 of its non-parallel sides, it passes also through the middles of the parallel sides $\{BC_1, CC_2\}$, hence also through the center K of κ. Analogous property holds also for the tangent E_2D_2 to κ'. It follows that line $\varepsilon = B'C'$ is the polar of K with respect to the hyperbola κ'.

(8) The pencil $D_1(OKA'A)$ consists of the sides of the triangle AD_1G_1, its median D_1E_1 and the parallel D_1O to its base, hence it is harmonic. It follows that $(O, K) \sim (A', A)$ are harmonic pairs.

In the case under consideration, in which κ is a parabola and κ' is a rectangular hyperbola, the center K of κ lies at infinity and, by (8), the conjugate O to it with respect to (A, A') must be coincident with the middle of $B'C'$ (see Figure

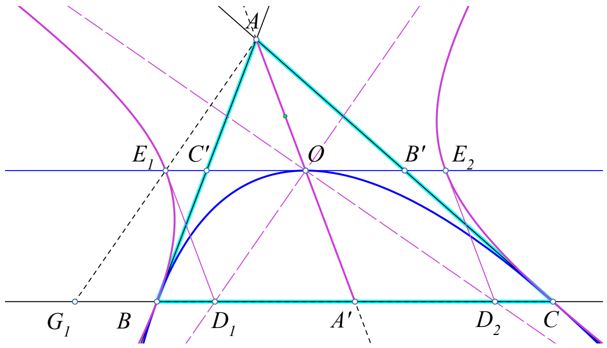

Figure 4. The case of parabola conjugate to a rectangular hyperbola

4). Taking line coordinates on BC with origin at G_1 and setting $\{x = G_1B, y = G_1D_1, BC = d\}$, the harmonicity relation becomes

$$\frac{G_1B}{G_1C} = -\frac{D_1B}{D_1C} \qquad \Leftrightarrow \qquad \frac{x}{x+d} = \frac{y-x}{x+d-y}.$$

Taking into account the parallelism of $\{AG_1, D_1J\}$ and the orthogonality of the triangle JD_1D_2, we find that $\{x, y\}$ satisfy also the equation

$$2x - 4y + d = 0.$$

From these equations, eliminating x, we deduce the equation $8y^2 - d^2 = 0$, which, because of $y = D_1A' = A'J = AA'/2$, is equivalent to the necessity condition of the theorem. The sufficiency of the condition is an easy reversing of the sequence of arguments and is left as an exercise.

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