Forum Geometricorum
Volume 18 (2018) 97–100.

FORUM GEOM
ISSN 1534-1178

# Remarks for The Twin Circles of Archimedes in a Skewed Arbelos by Okumura and Watanabe

Hiroshi Okumura and Saburou Saitoh

**Abstract**. From the viewpoint of the division by zero ($0/0 = 1/0 = z/0 = 0$) and the division by zero calculus, we will show some surprising new phenomena for geometrical properties with the concrete example which was given by the paper "The Twin Circles of Archimedes in a Skewed Arbelos" of H. Okumura and T. Watanabe, *Forum Geom.*, 4 (2004) 229–251.

## 1. Introduction

Let $V_z$ be the point with coordinates $(0,\ 2\sqrt{ab}/z)$ for real numbers $z$, and $a,\ b > 0$ in the plane. H. Okumura and M. Watanabe gave the following theorem in [7]:

**Theorem** (Theorem 7 of [7]). *The circle touching the circle $\alpha : (x-a)^2+y^2 = a^2$ and the circle $\beta : (x + b)^2 + y^2 = b^2$ at points different from the origin $O$ and passing through $V_{z_{\pm 1}}$ is represented by*

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{1}$$

*for a real number $z \neq \pm 1$. The common external tangents of $\alpha$ and $\beta$ can be expressed by the equations*

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0. \tag{2}$$

Anyhow the authors give the exact representation with a parameter of the general circles touching with two circles touching each other. The common external tangent may be considered a circle (as we know we can consider circles and lines as same ones in complex analysis or with the stereographic projection), however, they stated in the proof of the theorem that the common external tangents are obtained by the limiting $z \to \pm 1$. However, its logic will have a problem. Following our recent new concept of the division by zero calculus, we will consider the case $z = \pm 1$ for the singular points in the general parametric representation of the touching circles. Then, we will see interesting phenomena that we can discover a new circle in the context.

Publication Date: February 12, 2018. Communicating Editor: Paul Yiu.

## 2. The division by zero calculus

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n(z-a)^n, \tag{3}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{4}$$

(Here, as convention, we consider as $0^0 = 1$.)

For the correspondence (4) for the function $f(z)$, we will call it the *division by zero calculus*. By considering the derivatives in (3), we can define any order derivatives of the function $f$ at the singular point $a$.

We have considered our mathematics around an isolated singular point for analytic functions, however, we do not consider mathematics at the singular point itself. At the isolated singular point, we consider our mathematics with the limiting concept, however, the limiting values to the singular point and the value at the singular point of the function are different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. We will discuss the equation (1) from this viewpoint at the singular points $z = \pm 1$ that has a clear geometric meaning.

The division by zero ($0/0 = 1/0 = z/0 = 0$) is trivial and clear in the natural sense of the generalized division (fraction), since we know the Moore-Penrose generalized inverse for the elementary equation $az = b$. Therefore, the division by zero calculus above and its applications are important. See the references [10, 1, 5, 11, 2, 3, 6, 8, 4, 9] for the details and the related topics.

However, in this paper we do not need any information and results in the division by zero, we need only the definition (4) of the division by zero calculus.

## 3. Results

First, for $z = 1$ and $z = -1$, respectively by the division by zero calculus, we have from (1), surprisingly

$$x^2 + \frac{b-a}{2}x + y^2 \mp \sqrt{ab}y - ab = 0,$$

respectively.

Secondly, multiplying (1) by $(z^2 - 1)$, we immediately obtain surprisingly (2) for $z = 1$ and $z = -1$, respectively by the division by zero calculus.

In the usual way, when we consider the limiting $z \to \infty$ for (1), we obtain the trivial result of the point circle of the origin. However, the result may be obtained by the division by zero calculus at $w = 0$ by setting $w = 1/z$.

## 4. On the circle appeared

Let us consider the circle $\zeta$ expressed by

$$x^2 + \frac{b-a}{2}x + y^2 - \sqrt{ab}y - ab = 0.$$

Then $\zeta$ meets the circle $\alpha$ in two points

$$P_a\left(2r_A,\ 2r_A\sqrt{\frac{a}{b}}\right),\quad Q_a\left(\frac{2ab}{9a+b},\ -\frac{6a\sqrt{ab}}{9a+b}\right),$$

where $r_A = ab/(a+b)$ (see Figure 1). Also $\zeta$ meets the circle $\beta$ in points

$$P_b\left(-2r_A,\ 2r_A\sqrt{\frac{b}{a}}\right),\quad Q_b\left(\frac{-2ab}{a+9b},\ -\frac{6b\sqrt{ab}}{a+9b}\right).$$

Notice that $P_aP_b$ is the external common tangent of $\alpha$ and $\beta$ expressed by (2) with the minus sign. The lines $P_aQ_a$ and $P_bQ_b$ intersect at the point $R(0,\ -\sqrt{ab})$, which lies on the remaining external common tangent of $\alpha$ and $\beta$. Furthermore, the circle $\zeta$ is orthogonal to the circle with center $R$ passing through the origin.

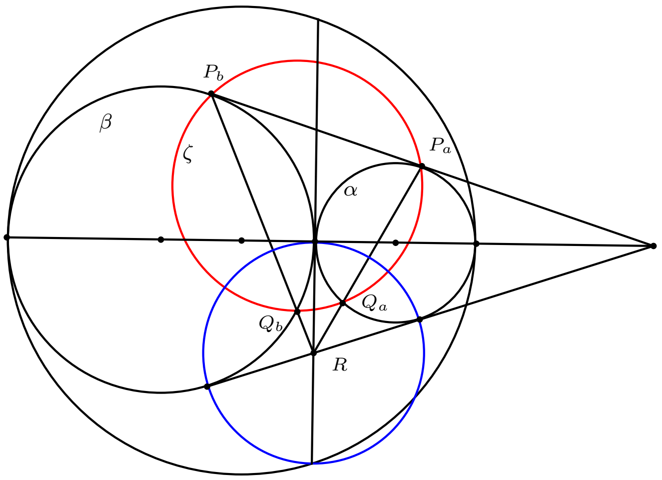


Figure 1.

## 5. Conclusion

By the division by zero calculus, we are able to obtain definite meaningful results simply. The new result of Section 4has never been expected thereby will have a special interest.

## References

[1] M. Kuroda, H. Michiwaki, S. Saitoh and M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, *Int. J. Appl. Math.*, 27 (2014) 191–198; DOI: 10.12732/ijam.v27i2.9.

[2] T. Matsuura and S. Saitoh, Matrices and division by zero $z/0 = 0$, *Advances in Linear Algebra & Matrix Theory*, 6 (2016) 51–58;
published Online June 2016 in SciRes. http://www.scirp.org/journal/alamt, http://dx.doi.org/10.4236/alamt.2016.62007.

[3] T. Matsuura and S. Saitoh, Division by zero calculus and singular integrals; submitted for publication.

[4] T. Matsuura, H. Michiwaki and S. Saitoh, $\log 0 = \log \infty = 0$ and applications, in *Differential and Difference Equations with Applications*, Springer Proceedings in Mathematics & Statistics.
[5] H. Michiwaki, S. Saitoh and M. Yamada, Reality of the division by zero $z/0 = 0$, *International J. of Applied Physics and Math.*, 6 (2015) 1–8.
http://www.ijapm.org/show-63-504-1.html.
[6] H. Michiwaki, H. Okumura and S. Saitoh, Division by zero $z/0 = 0$ in Euclidean Spaces, *International Journal of Mathematics and Computation*, 28 (2017), 1–16.
[7] H. Okumura and M. Watanabe, The twin circles of Archimedes in a skewed arbelos, *Forum Geom.*, 4 (2004) 229–251.
[8] H. Okumura, S. Saitoh and T. Matsuura, Relations of $0$ and $\infty$, *Journal of Technology and Social Science*, 1 (2017) 70–77.
[9] S. Pinelas and S. Saitoh, Division by zero calculus and differential equations, in *Differential and Difference Equations with Applications*, Springer Proceedings in Mathematics & Statistics.
[10] S. Saitoh, Generalized inversions of Hadamard and tensor products for matrices, *Advances in Linear Algebra & Matrix Theory*, 4 (2014) 87–95;
http://www.scirp.org/journal/ALAMT/.
[11] S. Saitoh, A reproducing kernel theory with some general applications, in
T. Qian/L. Rodino, (eds.): *Mathematical Analysis, Probability and Applications - Plenary Lectures: Isaac 2015, Macau, China*, Springer Proceedings in Mathematics and Statistics, 177 (2016) 151–182 (Springer).

Hiroshi Okumura: Maebashi Gunma 371-0123, Japan
*E-mail address*: hokmr@protonmail.com

Saburou Saitoh: Institute of Reproducing Kernels Kawauchi-cho, 5-1648-16, Kiryu 376-0041, Japan
*E-mail address*: kbdmm360@yahoo.com.jp