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On the Cyclic Quadrilaterals with the Same Varignon Parallelogram

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Abstract. The circumcenters and anticenters of the quadrilaterals cyclic $PQRS$ with the same Varignon parallelogram describe an hyperbola Γ. We propose the characterization of this hyperbola. The vertices of $PQRS$ one by one move also on hyperbolas which are the translations of hyperbola Γ.

1. Introduction

Let A, B, C, D be the midpoints of the sides PQ, QR, RS, SP of a quadrilateral $PQRS$. The quadrilateral $ABCD$ is called the *Varignon parallelogram* of $PQRS$. Note with θ the angle ABC, $\theta \in (0, \pi)$ and let $AB = 2q$, $BC = 2p$, where $p > 0$, $q > 0$ (Figure 1). Let O be the center of symmetry of the parallelogram $ABCD$. If the triplet (p, q, θ) is given and the parallelogram $ABCD$ is fixed, then there are an infinite number of quadrilaterals $PQRS$ with the same Varignon parallelogram $ABCD$ [1]. The construction of a such quadrilateral is the following: let X be an arbitrary point in the plan of parallelogram $ABCD$ and $X \notin AB$; construct the anticomplementary triangle of the triangle ABX and note its vertices with P_X, Q_X, R_X (let A be the midpoint of P_XQ_X, B the midpoint of Q_XR_X and X the midpoint of P_XR_X); the point S_X will be the symmetric of R_X with respect to C. Call $P_XQ_XR_XS_X$ the *quadrilateral generated by the points* A, B and X. Denote the set of this quadrilaterals by $\mathcal{G}_{ABX}$.

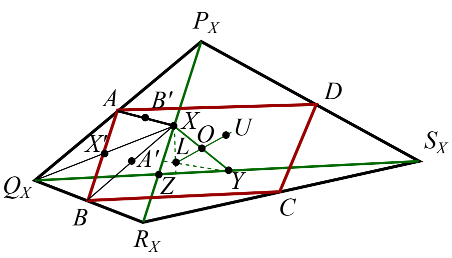

Figure 1

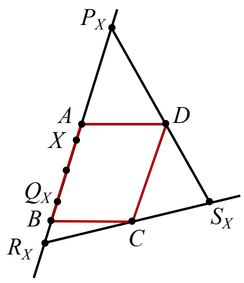

Figure 2

Let A', B', X' be the midpoints of the segments BX, XA, AB (Figure 1). The points P_X, Q_X, R_X are the symmetric of B, X, A with respect to the points B', X', A' respectively. From this property results an other construction of the quadrilateral $P_XQ_XR_XS_X$: consider for example the symmetric of X with respect to the midpoint of AB (i.e. the point Q_X); R_X will be the symmetric of Q_X with

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respect to B, S_X the symmetric of R_X with respect to C, P_X the symmetric of S_X with respect to D. With this method is possible the construction of the quadrilateral $P_XQ_XR_XS_X$ for the points $X \in AB$ (Figure 2).

In this paper we assume that the parallelogram $ABCD$ is fixed and for all points X from the plane of $ABCD$ the quadrilaterals $P_XQ_XR_XS_X$ have $ABCD$ as Varignon parallelogram. Denote this set by $\mathcal{G}_X$. Therefore

$$\mathcal{G}_X = \mathcal{G}_{ABX} \cup \mathcal{G}_{BCX} \cup \mathcal{G}_{CDX} \cup \mathcal{G}_{DAX}.$$

$\mathcal{G}_X$ is the set of all quadrilaterals $P_XQ_XR_XS_X$ generated by the point X.

Remark 1. If two quadrilaterals in $\mathcal{G}_X$ have a common vertex, then they coincide.

We pose the following question: *for what positions of the point X will the quadrilateral $P_XQ_XR_XS_X$ be cyclic*?

2. The set of quadrilaterals $P_XQ_XR_XS_X$

Attach to the parallelogram $ABCD$ an oblique coordinate system xOy with origin at O, x- and y-axes parallel to BC and AB respectively (Figure 3). The coordinates of the points A, B, C, D are $A = (-p, q)$, $B = (-p, -q)$, $C = (p, -q)$, $D = (p, q)$.

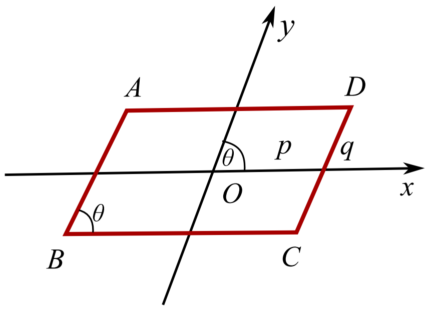

Figure 3

If $X = (\alpha, \beta)$, then

$$\begin{aligned}
2A' &= X + B = (\alpha - p,\ \beta - q),\\
2B' &= X + A = (\alpha - p,\ \beta + q),\\
2X' &= A + B = (-2p,\ 0),\\
P_X &= 2B' - B = X + A - B = (\alpha - p,\ \beta + q) - (-p, q) = (\alpha,\ \beta + 2q),\\
Q_X &= 2X' - X = A + B - X = (-2p,\ 0) - (\alpha,\ \beta) = (-\alpha - 2p,\ -\beta),\\
R_X &= 2A' - A = X + B - A = (\alpha - p,\ \beta - q) - (-p,\ q) = (\alpha,\ \beta - 2q).
\end{aligned}$$

If $S_X = 2C - R = (2p, -2q) - (\alpha, \beta - 2q) = (-\alpha + 2p, -\beta)$, then $2D = P_X + S_X$, i.e. D is the midpoint of segment P_XS_X. Consequently, $ABCD$ is the Varignon parallelogram of the quadrilateral $P_XQ_XR_XS_X$.

The point X is the midpoint of diagonal $P_X R_X$. Let Y be the midpoint of the other diagonal $Q_X S_X$. Since

$$Y = \frac{1}{2}(Q_X + S_X) = \frac{1}{2}(-\alpha - 2p,\ -\beta) + \frac{1}{2}(-\alpha + 2pm\ - \beta) = (-\alpha,\ -\beta),$$

the points X and Y are symmetric with respect to O (Figure 1).

Lemma 1. (a) $\mathcal{G}_{ABX} = \mathcal{G}_{CDX}$;
(b) $\mathcal{G}_{BCX} = \mathcal{G}_{DAX}$.

Proof. (a) Let $P_X Q_X R_X S_X \in \mathcal{G}_{ABX}$ and $P'_X Q'_X R'_X S'_X \in \mathcal{G}_{CDX}$. We have

$$S'_X = C + D - X = (p,\ -q) + (p,\ q) - (\alpha,\ \beta) = (-\alpha + 2p,\ -\beta) = S_X,$$

i.e. the quadrilaterals $P_X Q_X R_X S_X$ and $P'_X Q'_X R'_X S'_X$ coincide. □

Corollary 2. $\mathcal{G}_X = \mathcal{G}_{ABX} \cup \mathcal{G}_{BCX} = \mathcal{G}_{CDX} \cup \mathcal{G}_{DAX}$.

Lemma 3. *If X and Y are symmetric with respect to O, then*
(a) $\mathcal{G}_{ABX} = \mathcal{G}_{DAY}$;
(b) $\mathcal{G}_{BCX} = \mathcal{G}_{CDY}$.

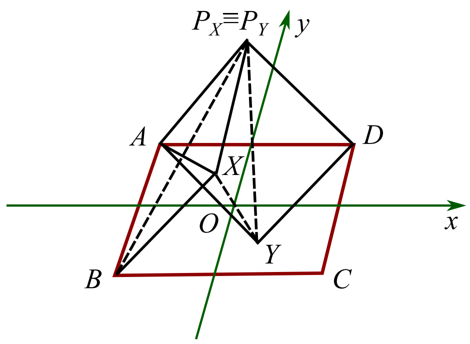

Figure 4

Proof. (a) Let $P_X Q_X R_X S_X \in \mathcal{G}_{ABX}$ and $PYQYRYSY \in \mathcal{G}_{DAY}$. We have

$$\begin{aligned} P_X &= \ X + A - B = (\alpha,\ \beta) + (-p,\ q) - (-p,\ -q) = (\alpha,\ \beta + 2q), \\ PY &= \ A + D - Y = (-p,\ q) + (p,\ q) - (-\alpha,\ -\beta) = (\alpha,\ \beta + 2q) = P_X, \end{aligned}$$

i.e. the quadrilaterals $P_X Q_X R_X S_X$ and $P_Y Q_Y R_Y S_Y$ coincide (Figure 4). □

Corollary 4. *If X and Y are symmetric with respect to O, then $\mathcal{G}_X = \mathcal{G}_Y$.*

Henceforth we denote the quadrilaterals $P_X Q_X R_X S_X$ more simply w by $(PQRS)_X$ or $PQRS$.

3. Special cases

I. If $p = q$, then $ABCD$ is rhombus. If $X \in BD$ (first bisector of xOy) then

$$P = (\alpha,\ \alpha + 2p), \quad Q = (-\alpha - 2p,\ -\alpha),$$
$$R = (\alpha,\ \alpha - 2p), \quad S = (-\alpha + 2p,\ -\alpha).$$

Since $\overrightarrow{PQ} = -2(\alpha + p,\ \alpha + p)$, $\overrightarrow{BD} = 2(p,\ p)$, $\overrightarrow{RS} = -2(\alpha - p,\ \alpha - p)$, results then $PQ \parallel BD \parallel RS$. In oblique coordinate system the length of segment EF determined by two points $E = (x_1, y_1)$ and $F = (x2,\ y2)$ is

$$EF = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\theta}.$$

In our case

$$PS^2 = 4(\alpha - p)^2 + 4(\alpha + p)^2 + 8(\alpha - p)(\alpha + p)\cos\theta = QR^2;$$

therefore, the quadrilateral $PQRS$ is an isosceles trapezoid, so it is cyclic (Figure 5).

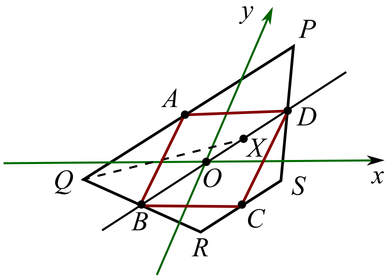

Figure 5

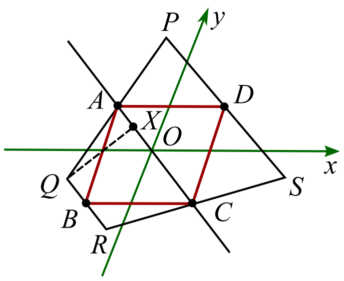

Figure 6

If $X \in AC$ (second bisector of xOy), then $\alpha = -\beta$ and $P = (\alpha,\ -\alpha + 2p)$, $Q = (-\alpha - 2p,\ \alpha)$, $R = (\alpha,\ -\alpha - 2p)$, $S = (-\alpha + 2p,\ \alpha)$. Since $\overrightarrow{PS} = -2(\alpha - p,\ -\alpha + p)$, $\overrightarrow{AC} = 2(p,\ -p)$, $\overrightarrow{RQ} = -2(\alpha + p,\ -\alpha - p)$, these give $PS \parallel AC \parallel RQ$ and

$$PQ^2 = 4(\alpha + p)^2 + 4(\alpha - p)^2 - (\alpha + p)(\alpha - p)\cos\theta = RS^2.$$

The quadrilateral $PQRS$ is also an isosceles trapezoid (Figure 6).

Conclusion: if $p = q$ and the point X is on the first or second bisector of the coordinate system xOy, then the quadrilateral $PQRS$ is cyclic.

II. If $p \neq q$ and $X \equiv D$, then $P = (p,\ 3q)$, $Q = (-3p,\ -q)$, $R = (p,\ -q) = C = S$, so the quadrilateral $PQRS$ will be a triangle (Figure 7).

If $X \notin \{A,\ B,\ C\}$, then the quadrilateral $PQRS$ turn into triangles, too, consequently $PQRS$ is cyclic. In the following we suppose that $p \neq q$.

III. If $X \equiv O$, then $P_O = (0,\ 2q)$, $Q_O = (-2p,\ 0)$, $R_O = (0,\ -2q)$, $S_O = (2p,\ 0)$, so the quadrilateral $P_OQ_OR_OS_O$ is a parallelogram which has $ABCD$ as Varignon parallelogram (Figure 8). Call $P_OQ_OR_OS_O$ the *zero parallelogram*.

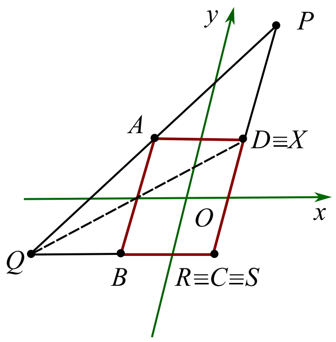

Figure 7

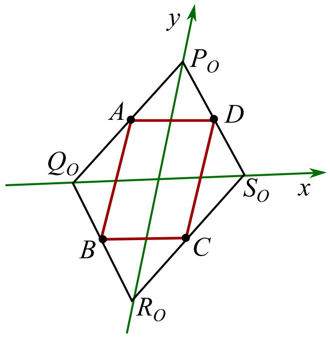

Figure 8

4. Conditions for a quadrilateral to be cyclic

Let $P,\ Q,\ R,\ S$ be the geometrical images of complex numbers z_P, z_Q, z_R, z_S.

Theorem 5. *The following statements are equivalent:*

(a) *The quadrilateral $PQRS$ is cyclic.*

(b) $\mathrm{Im}\left(\frac{z_P-z_R}{z_P-z_S} : \frac{z_Q-z_R}{z_Q-z_S}\right) = 0$*, where* $\mathrm{Im}z$ *is the imaginary part of the complex number z.*

(c) *The perpendicular bisectors of sides PQ, QR, RS, SP are concurrent in a point U (see [1] Theorem 4).*

(d) *There is a point V in the plane of $PQRS$ that $VP = VQ = VR = VS$.*

Theorem 6. *If $X \notin \{A,\ B,\ C,\ D\}$, then among the quadrilaterals with the same Varignon parallelogram $ABCD$ a quadrilateral $(PQRS)_X$ is cyclic if and only if*

$$\alpha^2 - \beta^2 = p^2 - q^2. \tag{1}$$

Proof. In this case

$$\begin{aligned} z_P &= \alpha + (\beta + 2q)i, \quad z_Q = -(\alpha + 2p) - \beta i, \\ z_R &= \alpha + (\beta - 2q)i, \quad z_S = -\alpha + 2p - \beta i. \end{aligned}$$

We have

$$\begin{aligned} z_P - z_R &= 4qi, \\ z_P - z_S &= 2[\alpha - p + (\beta + q)i], \\ z_Q - z_R &= -2[\alpha + p + (\beta - q)i], \\ z_Q - z_S &= -4p \end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Im}\left(\frac{z_P - z_R}{z_P - z_S} : \frac{z_Q - z_R}{z_Q - z_S}\right) = 0\\
\Longleftrightarrow\ &\operatorname{Im}\frac{4pqi}{[\alpha - p + (\beta + q)i][\alpha + p + (\beta - q)i]} = 0\\
\Longleftrightarrow\ &\operatorname{Im}\frac{4pq}{[\alpha - p + (\beta + q)i][\beta - q - (\alpha + p)i]} = 0\\
\Longleftrightarrow\ &\operatorname{Im}\frac{4pq[\alpha - p - (\beta + q)i][\beta - q + (\alpha + p)i]}{[(\alpha - p)^2 + (\beta + q)^2][(\beta - q)^2 + (\alpha + p)^2]} = 0\\
\Longleftrightarrow\ &\alpha^2 - p^2 - (\beta^2 - q^2) = 0\\
\Longleftrightarrow\ &\alpha^2 - \beta^2 = p^2 - q^2.
\end{aligned}$$

□

Remark 4.1. If we start from the anticomplementary triangle of triangle BCX or CDX or DAX we obtain the same condition (1).

Consider the equation $x^2 - y^2 = p^2 - q^2$, which represents a rectangular hyperbola [2]. Note this hyperbola with Γ. So the quadrilateral $PQRS$ is cyclic if and only if the point X, the midpoint of diagonal PR, is on the hyperbola Γ. Henceforth from among all quadrilaterals with the same Varignon parallelogram $ABCD$ we consider only the quadrilaterals cyclic.

Remark 4.2. The points $A,\ B,\ C,\ D$ and Y are on the hyperbola Γ.

In this paper we propose the characterization of hyperbola Γ. Denote by $U(R_u)$ the circle circumscribed to the quadrilateral $PQRS$ with center U and radius R_u.

5. The locus of the centers U

The circumcenter U of quadrilateral $PQRS$ is the symmetric of the anticenter L with respect to O (the center of symmetry of the parallelogram $ABCD$). The anticenter of $PQRS$ is the orthocenter of triangle XYZ, where Z is the point of intersection of diagonals PR and QS (Figure 1). In the oblique coordinate system xOy the equation of the line perpendicular to the line with equation $lx+my+n = 0$ and which through the point $(x',\ y')$ is

$$\begin{vmatrix} x & y & 1\\ x' & y' & 1\\ l - m\cos\theta & m - l\cos\theta & 0 \end{vmatrix} = 0.$$

The equation of XZ, respectively ZY , are $x = \alpha$, respectively $y = -\beta$. The equation of the altitudes of triangle XYZ from X, respectively Y, are

$$\begin{vmatrix} x & y & 1 \\ \alpha & \beta & 1 \\ -\cos\theta & 1 & 0 \end{vmatrix} = 0 \Longleftrightarrow x + y\cos\theta = \alpha + \beta\cos\theta,$$

$$\begin{vmatrix} x & y & 1 \\ -\alpha & -\beta & 1 \\ 1 & -\cos\theta & 0 \end{vmatrix} = 0 \Longleftrightarrow x\cos\theta + y = -(\alpha\cos\theta + \beta).$$

We determine the coordinates of the anticenter:

$$\Delta = \begin{vmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{vmatrix} = 1 - \cos^2\theta = \sin^2\theta, \quad \text{if } \theta \notin (0, \pi),$$

$$\Delta_x = \begin{vmatrix} \alpha + \beta\cos\theta & \\ -(\alpha\cos\theta + \beta) & 1 \end{vmatrix} = \alpha + 2\eta\cos\theta + \alpha\cos^2\theta,$$

$$\Delta_y = \begin{vmatrix} 1 & \alpha + \beta\cos\theta \\ \cos\theta & -(\alpha\cos\theta + \beta) \end{vmatrix} = -(\beta + 2\alpha\cos\theta + \beta\cos^2\theta).$$

Therefore, the coordinates of U are

$$x_U = -x_L = -\frac{\Delta_x}{\Delta} = -\frac{\alpha + 2\beta\cos\theta + \alpha\cos^2\theta}{\sin^2\theta} \quad \text{and}$$
$$y_U = -y_L = -\frac{\Delta_y}{\Delta} = \frac{\beta + 2\alpha\cos\theta + \beta\cos^2\theta}{\sin^2\theta}.$$

Theorem 7. *The centers U and the anticenters L of all quadrilaterals cyclic with the same Varignon parallelogram describe the same rectangular hyperbola Γ.*

Proof. Indeed,

$$\begin{aligned} x_U^2 - y_U^2 = &\ \frac{1}{\sin^4\theta}\left((\alpha + 2\beta\cos\theta + \alpha\cos^2\theta)^2 - (\beta + 2\alpha\cos\theta + \beta\cos^2\theta)^2\right) \\ = &\ \frac{1}{\sin^4\theta}\left((\alpha + 2\beta\cos\theta + \alpha\cos^2\theta - \beta - 2\alpha\cos\theta - \beta\cos^2\theta\right) \\ &\ \cdot\left(\alpha + 2\beta\cos\theta + \alpha\cos^2\theta + \beta + 2\alpha\cos\theta + \beta\cos^2\theta\right) \\ = &\ \frac{1}{\sin^4\theta}(\alpha - \beta)(1 - \cos\theta)^2(\alpha + \beta)(1 + \cos\theta)^2 \\ = &\ \alpha^2 - \beta^2 \\ = &\ p^2 - q^2. \end{aligned}$$

□

Remark 5.1. The Euler center and the anticenter of a cyclic quadrilateral coincide (see [1] Theorem 5), results that the Euler center is on the hyperbola Γ too.

6. The radius of circle $U(R_u)$

Theorem 8. *The radius of the circle $U(R_u)$ is*

$$\begin{aligned} R_u = &\ \frac{2}{\sin\theta}\sqrt{(\alpha+\beta\cos\theta)^2+q^2\sin^2\theta} \\ = &\ \frac{2}{\sin\theta}\sqrt{(\alpha\cos\theta+\beta)^2+p^2\sin^2\theta}. \end{aligned} \tag{2}$$

Proof. We have

$UP^2 = |z_P - z_U|^2 = (\alpha - x_U)^2 + (\beta+2q-y_U)^2 + 2(\alpha - x_U)(\beta+2q-y_U)\cos\theta,$

where

$$\begin{aligned} \alpha - xU = &\ \alpha + \frac{\alpha+2\beta\cos\theta+\alpha\cos^2\theta}{\sin^2\theta} = \frac{2(\alpha+\beta\cos\theta)}{\sin^2\theta} \quad \text{and} \\ \beta - y_U = &\ \beta - \frac{\beta+2\alpha\cos\theta+\beta\cos^2\theta}{\sin^2\theta} = -\frac{2(\alpha+\beta\cos\theta)\cos\theta}{\sin^2\theta}. \end{aligned}$$

Consequently

$$\begin{aligned} R_u^2 = UP^2 = &\ \frac{4}{\sin^4\theta}\Big\{(\alpha+\beta\cos\theta)^2+\big[q\sin^2\theta-(\alpha+\beta\cos\theta)\cos\theta\big]^2 \\ &\ +2(\alpha+\beta\cos\theta)\big[q\sin^2\theta-(\alpha+\beta\cos\theta)\cos\theta\big]\cos\theta\Big\} \\ = &\ \frac{4}{\sin^2\theta}\big((\alpha+\beta\cos\theta)^2+q^2\sin^2\theta\big) \\ = &\ UR^2. \end{aligned}$$

Analogously we obtain

$UQ^2 = |z_Q - z_U|^2 = (\alpha+2p+x_U)^2 + (\beta+y_U)^2 + 2(\alpha+2p+x_U)(\beta+y_U)\cos\theta,$

where

$$\begin{aligned} \alpha + x_U = &\ \alpha - \alpha + 2\beta\cos\theta + \alpha\cos 2\theta\sin^2\theta = -2(\alpha\cos\theta+\beta)\cos\theta\sin^2\theta, \\ \beta + y_U = &\ \beta + \beta + 2\alpha\cos\theta + \beta\cos 2\theta\sin^2\theta = 2(\alpha\cos\theta+\beta)\sin^2\theta \quad \text{and} \\ R_u^2 = UQ^2 = &\ \frac{4}{\sin^4\theta}\Big\{(\alpha\cos\theta+\beta)^2+\big[p\sin^2\theta-(\alpha\cos\theta+\beta)\cos\theta\big]^2 \\ &\ +2(\alpha\cos\theta+\beta)\big[p\sin^2\theta-(\alpha\cos\theta+\beta)\cos\theta\big]\cos\theta\Big\} \\ = &\ \frac{4}{\sin^2\theta}\big((\alpha\cos\theta+\beta)^2+p^2\sin^2\theta\big) \\ = &\ US^2. \end{aligned}$$

□

7. The axes and the foci of hyperbola Γ

The asymptotes of hyperbola Γ are the first and the second bisector of the coordinate system xOy (Figure 9). Note these asymptotes with h and h'. The axes of symmetry of Γ are the interior and exterior bisectors of the right angle formed by h and h'. Note these axes of symmetry with a and a' (let a be the transverse axis).

We propose to determine the coordinates of vertices A_1 and A_2 of Γ . First, we write the equation of the line A_1A_2.

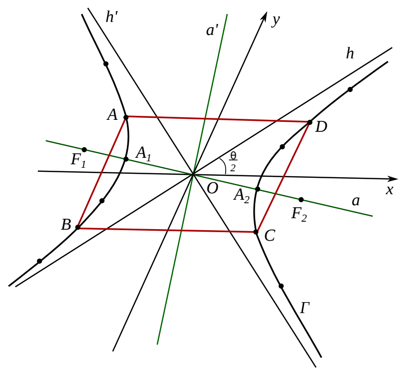

Figure 9

If a line OT forms an angle γ with the axe Ox, then the equation of OT in the oblique coordinates system xOy is $y = \frac{\sin\gamma}{\sin(\theta-\gamma)}x$. The line A_1A_2 forms an angle $\frac{\theta}{2} - \frac{\pi}{4}$ with the axe Ox. The equation of A_1A_2 is

$$y = \frac{\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}{\sin\left(\theta-\frac{\theta}{2}+\frac{\pi}{4}\right)}x = \frac{\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}{\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}x = \frac{\sin\frac{\theta}{2}-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}x = -\frac{\cos\theta}{1+\sin\theta}x.$$

With the notation $m = -\frac{\cos\theta}{1+\sin\theta}$, the equation of A_1A_2 is $y = mx$.

Theorem 9. *The major axis of the hyperbola* Γ *is*

$$A_1A_2 = 2\sqrt{|p^2-q^2|\sin\theta}. \tag{3}$$

Proof. We determine the coordinates of the vertices A_1 and A_2:

$$x^2 - m2x^2 = p^2 - q^2 \Longleftrightarrow x^2 = \frac{p^2-q^2}{1-m^2}.$$

The roots are

$$x_1 = -\sqrt{\frac{|p^2-q^2|}{1-m^2}} \quad \text{and} \quad x_2 = \sqrt{\frac{|p^2-q^2|}{1-m^2}},$$

since $1-m^2 = 1-\left(\frac{\cos\theta}{1+\sin\theta}\right)^2 = \frac{2\sin\theta}{1+\sin\theta} > 0$ if $\theta \in (0,\pi)$. Therefore

$$A_1 = (x_1,\ mx_1) = \left(-\sqrt{\frac{1+\sin\theta}{2\sin\theta}|p^2-q^2|}, \frac{\cos\theta}{1+\sin\theta}\sqrt{\frac{1+\sin\theta}{2\sin\theta}|p^2-q^2|}\right),$$

$$A_2 = (x_2,\ mx_2) = \left(\sqrt{\frac{1+\sin\theta}{2\sin\theta}|p^2-q^2|},\ -\frac{\cos\theta}{1+\sin\theta}\sqrt{\frac{1+\sin\theta}{2\sin\theta}|p^2-q^2|}\right).$$

We calculate the length of segments $OA_1 = OA_2$:

$$\begin{aligned} OA_1^2 = OA_2^2 &= \frac{p^2-q^2}{1-m^2} + m^2\frac{p^2-q^2}{1-m^2} + 2m\frac{p^2-q^2}{1-m^2}\cos\theta \\ &= \frac{p^2-q^2}{1-m^2}(1+m^2+2m\cos\theta) \\ &= \frac{p^2-q^2}{1-m^2}\left(2 - \frac{2\sin\theta}{1+\sin\theta} - \frac{2\cos^2\theta}{1+\sin\theta}\right) \\ &= (p^2-q^2)\sin\theta. \end{aligned}$$

It follows that $A_1A_2 = 2\cdot OA_1 = 2\sqrt{|p^2-q^2|\sin\theta}$. □

Theorem 10. *The coordinates of the foci F_1 and F_2 of the hyperbola Γ are*

$$F_1 = \left(-\sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|}, \frac{\cos\theta}{1+\sin\theta}\sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|}\right),$$

$$F_2 = \left(\sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|}, -\frac{\cos\theta}{1+\sin\theta}\sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|}\right).$$

Proof. Since $-F_1 = F_2 = (x,\ mx)$ and

$$OF_1 = OF_2 = \sqrt{2}\cdot OA_1 = \sqrt{2}\sqrt{|p^2-q^2|\sin\theta},$$

we have

$$\begin{aligned} x^2 + m^2x^2 + 2mx^2\cos\theta &= 2|p^2-q^2|\sin\theta \\ \iff (1+m^2+2m\cos\theta)x^2 &= 2|p^2-q^2|\sin\theta. \end{aligned}$$

Therefore,

$$\frac{2\sin^2\theta}{1+\sin\theta}x^2 = 2|p^2-q^2|\sin\theta \iff x_{1,2} = \sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|}.$$

□

8. The locus of the vertices P, Q, R, S

Theorem 11. *The locus of the vertices P, Q, R, S are the hyperbolas Γ_P, Γ_Q, Γ_R, Γ_S, the translations of hyperbola Γ by the vectors $(0,\ 2q)$, $(-2p,\ 0)$, $(0,\ -2q)$, $(2p,\ 0)$ respectively.*

Proof. The coordinates of the point P are $x = \alpha$ and $y = \beta + 2q$. Since the point $(\alpha,\ \beta)$ describes the hyperbola Γ, with equation $\alpha^2 - \beta^2 = p^2 - q^2$, the locus of P will be the hyperbola Γ_P with equation $x^2 - (y-2q)^2 = p^2 - q^2$. The hyperbola Γ_P is the translation of Γ by the vector $(0,\ 2q)$. The equations of hyperbolas Γ_P, $\Gamma_Q, \Gamma_R, \Gamma_S$ are

$$\begin{array}{rlcl} \Gamma_P: & x^2-(y-2q)^2 = p^2-q^2 & \iff & x^2-y^2+4qy-p^2-3q^2=0; \\ \Gamma_Q: & (x+2p)^2-y^2 = p^2-q^2 & \iff & x^2-y^2+4px+3p^2+q^2=0; \\ \Gamma_R: & x^2-(y+2q)^2 = p^2-q^2 & \iff & x^2-y^2-4qy-p^2-3q^2=0; \\ \Gamma_S: & (x-2p)^2-y^2 = p^2-q^2 & \iff & x^2-y^2-4px+3p^2+q^2=0. \end{array}$$

□

Remark 8.1. The centers of symmetry of hyperbolas Γ_P, Γ_Q, Γ_R, Γ_S are the vertices of zero parallelogram $P_O Q_O R_O S_O$.

9. Orthodiagonal quadrilaterals

If $\theta = \frac{\pi}{2}$, then the diagonals PR and QS are perpendiculars (Figure 10). In this case the coordinates of vertices A_1 and A_2 of the hyperbola Γ are $A_1 = \left(-\sqrt{|p^2 - q^2|},\ 0\right)$, $A_2 = \left(\sqrt{|p^2 - q^2|},\ 0\right)$, The coordinates of the foci are $F_1 = \left(-\sqrt{2}\sqrt{|p^2 - q^2|},\ 0\right)$, $F_2 = \left(\sqrt{2}\sqrt{|p^2 - q^2|},\ 0\right)$.

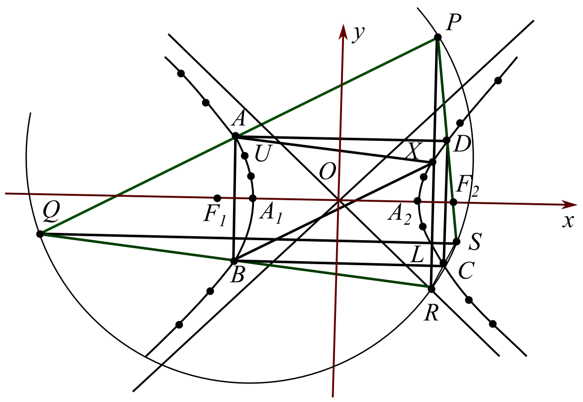

Figure 10

The center of circle circumscribed to the quadrilateral $PQRS$ is $U = (-\alpha,\ \beta)$, which is the symmetric of anticenter $(\alpha,\ -\beta)$, the point of intersection of diagonals PR and QS. In conclusion, in case of orthodiagonal quadrilaterals the point of intersection of its diagonals is also on the hyperbola Γ.

An open question is the geometric determination of the vertices and the foci of hyperbola Γ. Further it remains untackled if among the quadrilateral cyclic $(PQRS)_X$ could there be one with minimum area?

References

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