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An Extension of Miquel's Six-Circles Theorem

Gábor Gévay

Abstract. We extend the classical theorem of Miquel from 6 to $2n$ circles ($n \geq 3$). As a by-product of the proof of our theorem, we obtain the nice corollary that the product of the $2n$ cross ratios of the quadruples of points determining the circles is equal to 1. Moreover, the theorem can also be formulated in an equivalent form, which extends Miquel's Triangle Theorem to an arbitrary n-sided polygon.

Miquel's Six-Circles Theorem is a well known old theorem [5]. In Johnson's book [4], it is formulated in the following form.

Theorem 1 (Miquel)**.** *If the circles $A_1A_2B_3$, $A_2A_3B_1$, $A_3A_1B_2$ are concurrent at a point O, the circles $A_1B_2B_3$, $A_2B_3B_1$, $A_3B_1B_2$ are concurrent at a point P.*

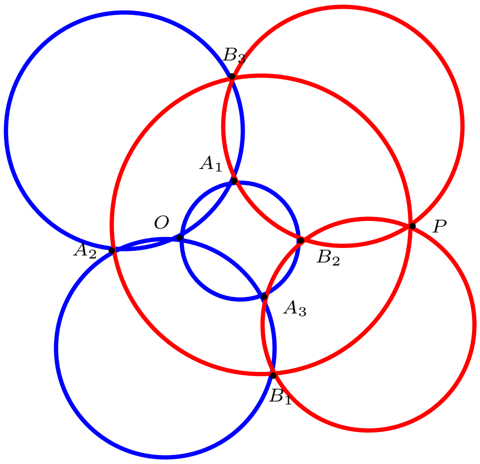

Figure 1. Miquel's Six-Circles Theorem

This formulation motivates the following extension.

Theorem 2. *Let $n \geq 3$ be an integer. Suppose that the circles $A_1B_1A_2$, $A_2B_2A_3$, $\dots$, $A_nB_nA_1$ are concurrent at a point O, and the circles $B_1A_2B_2$, $B_2A_3B_3$, $\dots$, $B_{n-1}A_nB_n$ are concurrent at a point O'. Then the circle $B_nA_1B_1$ also passes through O'.*

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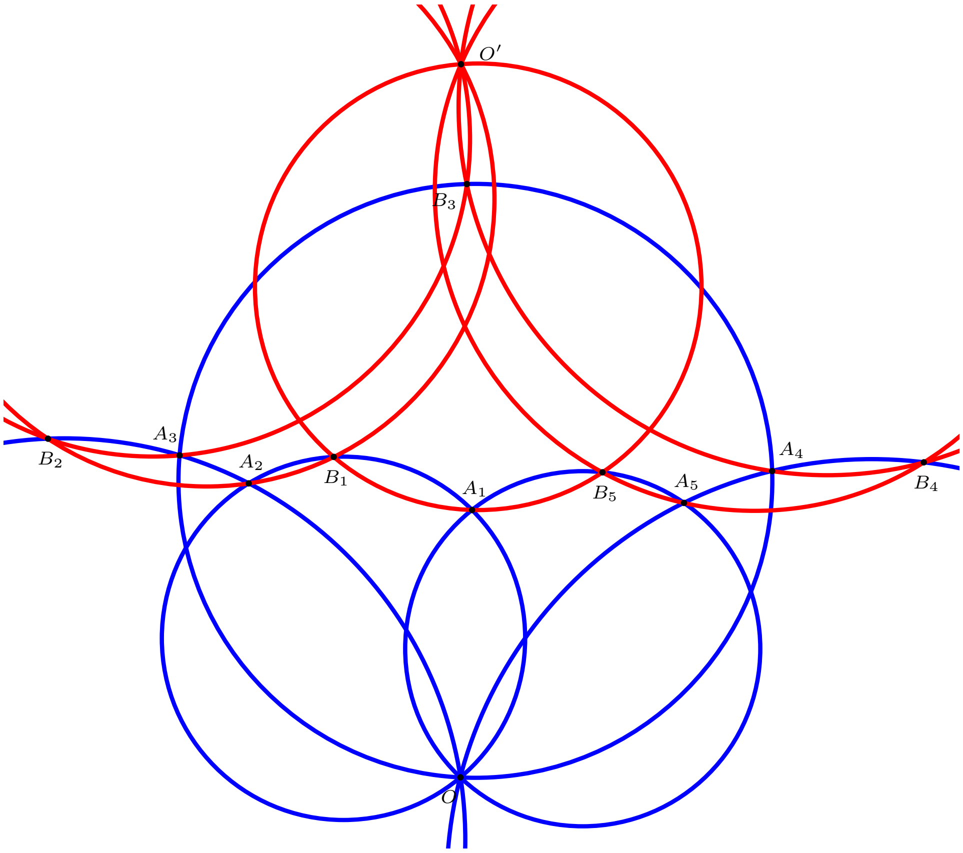

Figure 2. Example for Theorem 2 with $n = 5$

It is directly seen that for $n = 3$ it returns Miquel's original theorem; hence the latter theorem is an extension of the former, indeed.

In the proof, we shall use complex cross ratios. For a point X, we denote its *affix*, i.e. the corresponding complex number, by the same lowercase letter x. Given an ordered quadruple of points (X, Y, Z, W), we assign to it the cross ratio in the following form:

$$\frac{x-y}{x-w} : \frac{z-y}{z-w}.$$

We note that this is slightly different from the standard way used in the literature [3, 6], which is motivated by the particular arrangement of the points in our case; this choice is justified by the lemma given below. We shall need this lemma in the proof of our theorem.

Lemma 3. *Four points lie on the same circle if and only if their cross ratio is real.*

This is a well known result, see e.g. [3, 6]. Note that it is valid on the inversive plane, i.e. on the completed Euclidean plane $\mathbb{E}^2 \cup \{\infty\}$, where ∞ is the (unique) point at infinity [1]; the circles are meant in a generalized sense, i.e. they can also be (straight) lines.

Proof of Theorem 2. For $i \in \{1, \dots, n\}$, we denote the cross ratio of the quadruple (O, A_i, B_i, A_{i+1}) and $(O', B_i, A_{i+1}, B_{i+1})$ by γ_i and γ_i', respectively, where the indices are taken modulo n. Using the corresponding affixes, these cross ratios can be written as follows:

$$\gamma_i = \frac{o - a_i}{o - a_{i+1}} : \frac{b_i - a_i}{b_i - a_{i+1}} \text{ and } \gamma_i' = \frac{o' - b_i}{o' - b_{i+1}} : \frac{a_{i+1} - b_i}{a_{i+1} - b_{i+1}}, \tag{1}$$

respectively. Form the product

$$\begin{aligned}
\prod_{i=1}^{n} \gamma_i &= \left(\frac{o - a_1}{o - a_2} : \frac{b_1 - a_1}{b_1 - a_2}\right)\left(\frac{o - a_2}{o - a_3} : \frac{b_2 - a_2}{b_2 - a_3}\right) \cdots \left(\frac{o - a_n}{o - a_1} : \frac{b_n - a_n}{b_n - a_1}\right) \\
&= \left(1 : \frac{(b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)}{(b_1 - a_2)(b_2 - a_3)\cdots(b_n - a_1)}\right) \\
&= \frac{(b_1 - a_2)(b_2 - a_3)\cdots(b_n - a_1)}{(b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)} \\
&= \frac{(b_1 - a_2)(b_2 - a_3)\cdots(b_{n-1} - a_n)(b_n - a_1)}{(b_1 - a_1)(b_2 - a_2)\cdots(b_{n-1} - a_{n-1})(b_n - a_n)}.
\end{aligned} \tag{2}$$

Likewise, form the product

$$\begin{aligned}
\prod_{i=1}^{n-1} \gamma_i' &= \left(\frac{o' - b_1}{o' - b_2} : \frac{a_2 - b_1}{a_2 - b_2}\right)\left(\frac{o' - b_2}{o' - b_3} : \frac{a_3 - b_2}{a_3 - b_3}\right) \cdots \left(\frac{o' - b_{n-1}}{o' - b_n} : \frac{a_n - b_{n-1}}{a_n - b_n}\right) \\
&= \frac{o' - b_1}{o' - b_n} : \frac{(a_2 - b_1)(a_3 - b_2)\cdots(a_n - b_{n-1})}{(a_2 - b_2)(a_3 - b_3)\cdots(a_n - b_n)} \\
&= \frac{o' - b_1}{o' - b_n} : \frac{(b_1 - a_2)(b_2 - a_3)\cdots(b_{n-1} - a_n)}{(b_2 - a_2)(b_3 - a_3)\cdots(b_n - a_n)}.
\end{aligned} \tag{3}$$

Again we take the product $\left(\prod_{i=1}^{n} \gamma_i\right)\left(\prod_{i=1}^{n-1} \gamma_i'\right)$:

$$\begin{aligned}
&\frac{(b_1 - a_2)(b_2 - a_3)\cdots(b_{n-1} - a_n)(b_n - a_1)}{(b_1 - a_1)(b_2 - a_2)\cdots(b_{n-1} - a_{n-1})(b_n - a_n)} \\
&\quad \cdot \left(\frac{o' - b_1}{o' - b_n} : \frac{(b_1 - a_2)(b_2 - a_3)\cdots(b_{n-1} - a_n)}{(b_2 - a_2)(b_3 - a_3)\cdots(b_n - a_n)}\right) \\
=& \frac{(b_1 - a_2)(b_2 - a_3)\cdots(b_{n-1} - a_n)(b_n - a_1)}{(b_1 - a_1)(b_2 - a_2)\cdots(b_{n-1} - a_{n-1})(b_n - a_n)} \\
&\quad \cdot \frac{o' - b_1}{o' - b_n} \cdot \frac{(b_2 - a_2)(b_3 - a_3)\cdots(b_n - a_n)}{(b_1 - a_2)(b_2 - a_3)\cdots(b_{n-1} - a_n)} \\
=& \frac{b_n - a_1}{b_1 - a_1} \cdot \frac{o' - b_1}{o' - b_n} = \frac{b_n - a_1}{b_1 - a_1} : \frac{o' - b_n}{o' - b_1} = \frac{a_1 - b_n}{a_1 - b_1} : \frac{o' - b_n}{o' - b_1},
\end{aligned} \tag{4}$$

which is the reciprocal of the cross ratio of the quadruple (O', B_n, A_1, B_1).

Considering the condition of our theorem, and using Lemma 3, we see that the cross ratios $\gamma_1, \dots, \gamma_n$ and $\gamma_1', \dots, \gamma_{n-1}'$ are real. Hence the products (2) and (3) are also real, and so is (4). It follows, again by the lemma, that the quadruple (O', B_n, A_1, B_1) is cyclic. $\square$

As a consequence of the proof, we obtain the following corollary.

Corollary 4. *For the arrangement of points and circles described in Theorem 2, the following relation holds:*

$$\prod_{i=1}^{n} \gamma_i \gamma_i' = 1.$$

Recall that Miquel's Six-Circles Theorem has an equivalent version which is called Miquel's Triangle Theorem (it is also called the Pivot Theorem [1]; it also has interesting relationships in triangle geometry, see [7]). The two theorems are equivalent in terms of inversive geometry; indeed, the equivalence is based on the fact that choosing any point of intersection X of the circles, and applying an inversion with center X, the circles meeting at X transform to the side lines of a triangle.

Theorem 5 (Miquel)**.** *If $A_1A_2A_3$ is a triangle and B_1, B_2, B_3 are any three points on the lines A_2A_3, A_3A_1, A_1A_2, respectively, then the three circles $A_1B_2B_3$, $A_2B_3B_1$, $A_3B_1B_2$ have a common point.*

Likewise, Theorem 2 can also be given in the following equivalent form which, on the other hand, is an extension of Miquel's Triangle Theorem.

Theorem 6. *For an integer $n \geq 3$, let $A_1 \ldots A_n$ be a polygon, and let $B_1, \ldots,$ B_n be any points on the lines $A_1A_2, \ldots, A_nA_1$, respectively. Suppose that the circles $B_1A_2B_2, \ldots, B_{n-1}A_nB_n$ meet in the point P. Then the circle $B_nA_1B_1$ also passes through the point P.*

Note that in this extended case the equivalence is guaranteed by an inversion with center O.

Finally, we note that by a very recent result, Miquel's Six-Circles Theorem can also be extended in a different direction, as follows.

Theorem 7 ([2])**.** *Let α and β be two circles. Let $n > 2$ be an even number, and take the points $A_1, \ldots, A_n$ on α, and $B_1, \ldots, B_n$ on β, such that each quadruple $(A_1, B_1, A_2, B_2), \ldots, (A_{n-1}, B_{n-1}, A_n, B_n)$ is cyclic. Then the quadruple (A_n, B_n, A_1, B_1) is also cyclic.*

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Gábor Gévay: Bolyai Institute, University of Szeged, Szeged, Hungary
E-mail address: `gevay@math.u-szeged.hu`