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A Curvature Invariant Inspired by Leonhard Euler's Inequality $R \geq 2r$

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Abstract. It is of major interest to point out natural connections between the geometry of triangles and various other areas of mathematics. In the present work we show how Euler's classical inequality between circumradius and inradius inspires, by using a duality between triangle geometry and three-dimensional hypersurfaces lying in $\mathbb{R}^4$, the definition of a curvature invariant. We investigate this invariant by relating it to other known curvature invariants.

1. A historical motivation

Leonhard Euler believed that one could not define a good measure of curvature for surfaces. He wrote in 1760 that [5]: "la question sur la courbure des surfaces n'est pas susceptible d'une réponse simple, mais elle exige à la fois une infinité de déterminations." Euler's main objection to the very idea of the curvature of surfaces was that if one can approximate a planar curve with a circle (as Isaac Newton described), then for a surface it would be impossible to perform a similar construction. For example, Euler wrote in his investigation, what would we do with a saddle surface if we intend to find the sphere that best approximates it at a saddle point? On which side of the saddle should an approximating sphere lie? There are two choices and both appear legitimate. Today we see that Euler's counterexample corresponds to the case of a surface with negative Gaussian curvature at a given point, but in Euler's period no definition of curvature for a surface was available. Since Leonhard Euler contributed so much to the knowledge of mathematics in his time, one might expect that he was responsible for introducing and investigating a measure for the curvature of surfaces. However, this is not what happened. Instead, Leonhard Euler wrote a profound paper in which he obtained a well-known theorem about normal sections and stopped right there.

Moreover, Euler's paper [5] was extremely influential in the mathematical world of his times. When Jean-Baptiste Meusnier came to see his mentor, Gaspard Monge, for the first time, Monge instructed him to think of a problem related to the curvature of curves lying on a surface. Monge asked Meusnier to study Euler's paper [5], yet before reading it, in a stroke of genius the following night, Meusiner

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proved a theorem that today bears his name, and he did so by using a different geometric idea than Euler's. Decades afterwards, C.F. Gauss [7] introduced a measure for the curvature of surfaces that was complemented by the very inspired Sophie Germain [8], who introduced the mean curvature.

Perhaps the most direct way to explain the two curvature invariants for surfaces is the following. Consider a smooth surface S lying in $\mathbb{R}^3$, and an arbitrary point $P \in S$. Consider N_P, the normal to the surface at P, and the family of all planes passing through P that contain the line through P with the same direction as N_P. These planes yield a family of curves on S called *normal sections*. We now determine the curvature $\kappa(P)$ of the normal sections, viewed as planar curves. Then $\kappa(P)$ has a maximum, denoted κ_1, and a minimum, denoted κ_2. The curvatures κ_1 and κ_2 are called the *principal curvatures*. Using these principal curvatures, one may define the Gaussian curvature $K(P)$ as, $K(P) = \kappa_1(P) \cdot \kappa_2(P)$, and the mean curvature $H(P)$ as, $H(P) = \frac{1}{2}\left[\kappa_1(P) + \kappa_2(P)\right]$. Note that in the 18th century these constructions were not possible. They are related in a very profound way to the birth of non-Euclidean geometry and the revolution that took place in geometry at the beginning of the 19th century.

Taking into account this important historical context, we would like to investigate the following question. Was it possible for Leonhard Euler to have determined himself a curvature invariant, perhaps a concept for which all the algebra that he needed was in place, but in which it was just a matter of interpretation to define it? We are here in the territory of speculations, of alternative history, and only our surprise while we read and re-read Euler's original works leads us to such a question. However, this question makes a lot of sense as Euler had many contributions in geometry - from plane Euclidean geometry to the geometry of surfaces - and one of them in particular could have been tied to his investigations on curvature. Hence, was there any geometric property that was actually obtained by Euler which leads us to a property related to curvature?

2. Notations in the geometry of hypersurfaces

In order better approach the problem, we turn our attention to some classical ideas from the differential geometry of smooth hypersurfaces, a concept that was extensively explored after the second half of the 19th century. Let $\sigma : U \subset \mathbb{R}^3 \to \mathbb{R}^4$ be a hypersurface given by the smooth map σ and let p be a point on the hypersurface. Denote $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$, for all k from 1 to 3. Consider $\{\sigma_1(p), \sigma_2(p), \sigma_3(p), N(p)\}$, the orthonormal Gauss frame of the hypersurface, where N denotes the normal vector field to the hypersurface at every point.

The quantities similar to κ_1 and κ_2 in the geometry of surfaces are the principal curvatures, denoted λ_1, λ_2, and λ_3. They are introduced as the eigenvalues of the so-called Weingarten linear map, as we will describe below. Since our discussion is focused on the curvature quantities of three-dimensional smooth hypersurfaces in $\mathbb{R}^4$, we start by introducing these quantities. Similar to the geometry of surfaces, the curvature invariants in higher dimensions can also be described in terms of the

principal curvatures. The mean curvature at the point p is

$$H(p) = \frac{1}{3}[\lambda_1(p) + \lambda_2(p) + \lambda_3(p)],$$

and the Gauss-Kronecker curvature is

$$K(p) = \lambda_1(p)\lambda_2(p)\lambda_3(p).$$

In Riemannian geometry, a third important curvature quantity is the scalar curvature ([2], p.19) denoted by $scal(p)$, which intuitively sums up all the sectional curvatures on all the faces of the trihedron formed by the tangent vectors in the Gauss frame [4]:

$$scal(p) = \sec(\sigma_1 \wedge \sigma_2) + \sec(\sigma_2 \wedge \sigma_3) + \sec(\sigma_3 \wedge \sigma_1) = \lambda_1\lambda_2 + \lambda_3\lambda_1 + \lambda_2\lambda_3.$$

The last equality is due to the Gauss' equation of the hypersurface $\sigma(U)$ in the ambient space $\mathbb{R}^4$ endowed with the Euclidean metric.

To provide a brief description of how the Weingarten map is computed, we will review its construction here. Denote by $g_{ij}(p)$ the coefficients of the first fundamental form and by $h_{ij}(p)$ the coefficients of the second fundamental form. They are the outcome of the following dot products:

$$g_{ij}(p) = \langle \sigma_i(p), \sigma_j(p) \rangle, \qquad h_{ij}(p) = \langle N(p), \sigma_{ij}(p) \rangle.$$

The quantities $\lambda_1, \lambda_2, \lambda_3$ are always real; here we explain why. The Weingarten map $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \to T_{\sigma(p)}\sigma$ is linear. Denote by $h^i_j(p)_{1 \leq i,j \leq 3}$ the matrix associated to the Weingarten map, that is:

$$L_p(\sigma_i(p)) = \sum_{k=1}^{3} h^k_i(p)\sigma_k(p),$$

Weingarten's operator is self-adjoint, which implies that the roots of the algebraic equation

$$\det(h^i_j(p) - \lambda(p)\delta^i_j) = 0$$

are all real, at every point p of the hypersurface.

In the last decades several very interesting inequalities with geometric invariants have been obtained and investigated by many authors, see e.g. [2] for a thorough presentation of these developments.

3. A duality in triangle geometry

After this brief presentation of a few concepts pertaining to differential geometry, we will turn our attention to the classical geometry of a triangle. The duality described here is discussed also in [11]. Consider a triangle ΔABC in the Euclidean plane, with sides of lengths a, b, c.

Consider three arbitrary real numbers $x, y, z > 0$. By using the substitutions $a = y+z$, $b = z+x$, and $c = x+y$, one can immediately see that there exists a non-degenerate triangle in the Euclidean plane with sides of lengths a, b, c. The reason why this is true is that $a+b = x+y+2z > x+y = c$, hence the triangle inequality is always satisfied. Similarly for the other two triangle inequalities involving a, b, c.

The converse of this claim is also true. Namely, if a, b, c are the sides of a triangle in the Euclidean plane, then the system given through the equations $a = y+z$, $b = z+x$, and $c = x+y$, has a unique solution $x = s-a$, $y = s-b$, $z = s-c$, where we denote by s the semiperimeter of the triangle, namely $s = \frac{1}{2}(a+b+c)$.

Some classical facts in advanced Euclidean geometry can be proved by substituting a, b, c, instead of the variables x, y, z,. This technique is called Ravi substitution.

We can view these three positive real numbers x, y, z in a different way. We could view them as principal curvatures in the geometry of a three-dimensional smooth hypersurface lying in the four dimensional Euclidean space. To every triangle with sides a, b, c there corresponds a triple $x, y, z > 0$, and these numbers can be interpreted as principal curvatures, given pointwise, at some point p lying on the hypersurface. We call this correspondence a Ravi transformation and we would like to see whether this duality leads us to some interesting geometric interpretations of facts from triangle geometry into the geometry of hypersurfaces into $\mathbb{R}^4$ endowed with the canonical Euclidean product.

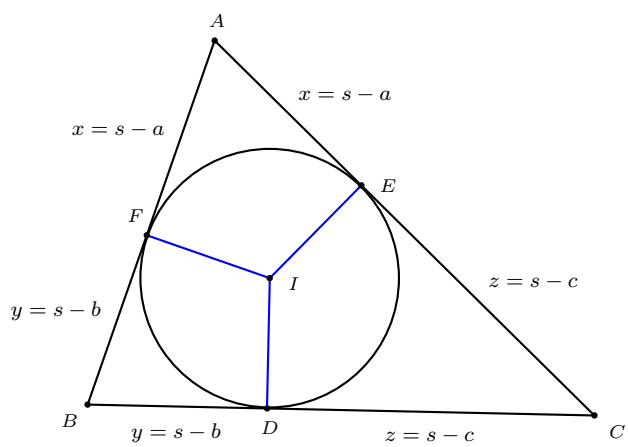

Figure 1

To better see how we use Ravi's substitutions, we recall here a few useful formulae for a triangle ΔABC lying in the Euclidean plane. Denote the area of the triangle by $\mathcal{A}$, its circumradius by R , its inradius by r, its semiperimeter by s, and its perimeter by $\mathcal{P}$. Then Heron's formula is $\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{xyz(x+y+z)}$. The inradius can be obtained as

$$r = \frac{\mathcal{A}}{s} = \sqrt{\frac{xyz}{x+y+z}},$$

and the circumradius can be obtained as

$$R = \frac{abc}{4\mathcal{A}} = \frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}}.$$

Euler's inequality was originally obtained in 1763 (see [6]) when he tried to construct a triangle by giving its incenter, its orthocenter, its barycenter and its circumcenter. That investigation generated several important consequences and ushered a new era in the investigations on the geometry of the triangle. Euler's inequality states that in any triangle in the Euclidean plane, the circumradius and the inradius satisfy $R \geq 2r$.

For a direct proof, using the formulae derived above, we note that by using the Ravi substitutions we need to show that the following holds:

$$\frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}} \geq 2\sqrt{\frac{xyz}{x+y+z}},$$

or, by a direct computation:

$$(y+z)(z+x)(x+y) \geq 8xyz.$$

This last inequality holds true since $x + y \geq 2\sqrt{xy}$, and two other similar inequalities, multiplied term by term, conclude the argument. Actually we notice that the inequality $(y+z)(z+x)(x+y) \geq 8xyz$ is equivalent, via Ravi transformation, to Euler's inequality $R \geq 2r$.

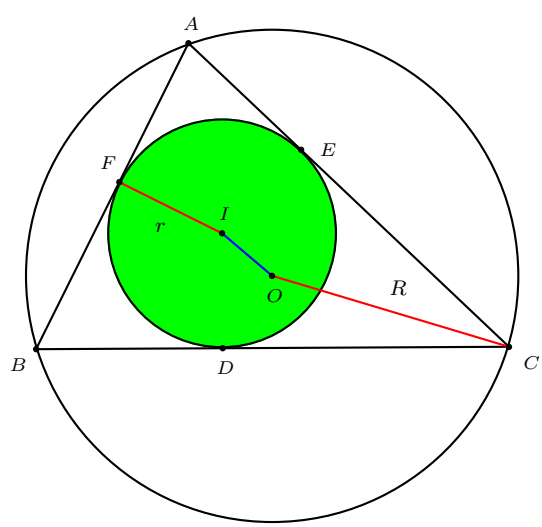

Figure 2

It is natural to consider a duality between a triangle ΔABC with lengths of sides a, b, c, and corresponding $x = s - a$, $y = s - b$, $z = s - c$, and a convex three-dimensional smooth hypersurface such that a point p lies on it, and the principal

curvatures stand for $\lambda_1(p) = x$, $\lambda_2(p) = y$, $\lambda_3(p) = z$. The convexity condition is needed to insure that $x, y, z > 0$ in an open neighborhood around p. For further details about this idea see [11].

4. Constructing a curvature invariant inspired by Euler's inequality

We recall here a class of curvature invariants introduced in [9], namely the sectional absolute mean curvatures defined by

$$\bar{H}_{ij}(p) = \frac{|\lambda_i(p)| + |\lambda_j(p)|}{2}.$$

By using the same method as above Euler's inequality, $R \geq 2r$, transforms, as we have seen before, into

$$(y+z)(z+x)(x+y) \geq 8xyz,$$

which admits the interpretation in terms of curvature invariants as follows:

$$\bar{H}_{12} \cdot \bar{H}_{23} \cdot \bar{H}_{31} \geq K.$$

This inequality between the curvature invariants of a three dimensional smooth hypersurface in $\mathbb{R}^4$ was obtained and investigated in [9]. We recall here the following.

Definition. [3] Let $\sigma : U \subset \mathbb{R}^3 \to \mathbb{R}^4$ be a hypersurface given by the smooth map σ. The point p on the hypersurface is called *absolutely umbilical* if all the principal curvatures satisfy $|k_1| = |k_2| = |k_3|$. If all the points of a hypersurface are absolutely umbilical, then the hypersurface is called absolutely umbilical.

In conclusion, a curvature invariant inspired by this result obtained by Leonhard Euler is

Definition.

$$E(p) = \bar{H}_{12} \cdot \bar{H}_{23} \cdot \bar{H}_{31} = \frac{1}{8}(y+z)(z+x)(x+y).$$

Building on the development from [1], in [9] the following is proved.

Theorem 1. *Let $M^3 \subset \mathbb{R}^4$ be a smooth hypersurface and k_1, k_2, k_3 be its principal curvatures in the ambient space $\mathbb{R}^4$ endowed with the canonical metric. Let $p \in M$ be an arbitrary point. Denote by A the amalgamatic curvature (for its definition see* [3, 1]*), by H the mean curvature, $\bar{H}$ the absolute mean curvature, and K the Gauss-Kronecker curvature. Introduce the sectional absolute mean curvatures defined by*

$$\bar{H}_{ij}(p) = \frac{|k_i(p)| + |k_j(p)|}{2}.$$

Then the following inequalities hold true at every point $p \in M$:

$$A \cdot \bar{H}_{12} \cdot \bar{H}_{23} \cdot \bar{H}_{13} \geq \bar{H} \cdot |K| \geq A \cdot |K|.$$

Equality holds for absolutely umbilical points.

In conclusion, the inequality in the theorem can be written as follows.

Corollary 2.

$$A \cdot E \geq \bar{H} \cdot |K| \geq A \cdot |K|.$$

This relation naturally relates the amalgamatic curvature, the Gaussian curvature, and the curvature invariant E, which we introduced inspired by Euler's inequality for triangles in Euclidean geometry, $R \geq 2r$. In particular, note that we have the following.

Corollary 3.

$$E(p) \geq |K(p)| \tag{1}$$

at any point p of $M^3 \subset \mathbb{R}^4$. The equality holds at a point p if and only if the point is absolutely umbilical.

It may be of interest to note that this inequality may be viewed as an extension of the inequality $H^2(p) \geq K(p)$ from the geometry of surfaces.

We conclude this section by pointing out that all the algebra was definitely in place in Euler's times, and the only new part was the geometric interpretation in terms of curvature invariants. This vision was greatly enhanced after the major revisitation of curvature invariants described in Bang-Yen Chen's comprehensive monograph [2], which best represents a research direction with many developments (see e.g. [10, 12], among many other works), which inspires also the context of our present investigation.

5. Hypersurfaces of dimension n lying in the Euclidean space $\mathbb{R}^{n+1}$

In this section we inquire whether it is possible to extend the inequality (1) to hypersurfaces in the general case.

To establish our notations, let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. Let p be a point on the hypersurface. Denote $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$, for all k from 1 to n. Consider $\{\sigma_1(p), \sigma_2(p), ..., \sigma_n(p), N(p)\}$, the Gauss frame of the hypersurface, where N denotes the normal vector field. We denote by $g_{ij}(p)$ the coefficients of the first fundamental form and by $h_{ij}(p)$ the coefficients of the second fundamental form. Then we have

$$g_{ij}(p) = \langle \sigma_i(p), \sigma_j(p) \rangle, \qquad h_{ij}(p) = \langle N(p), \sigma_{ij}(p) \rangle.$$

The Weingarten map $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \to T_{\sigma(p)}\sigma$ is linear. Denote by $(h^i_j(p))_{1 \leq i,j \leq n}$ the matrix associated to Weingarten's map, that is:

$$L_p(\sigma_i(p)) = h^k_i(p)\sigma_k(p),$$

where the repeated index and upper script above indicates Einstein's summation convention. Weingerten's operator is self-adjoint, which implies that the roots of the algebraic equation

$$\det(h^i_j(p) - \lambda(p)\delta^i_j) = 0$$

are real. The eigenvalues of Weingarten's linear map are called principal curvatures of the hypersurface. They are the roots $k_1(p), k_2(p), ..., k_n(p)$ of this algebraic

equation. The mean curvature at the point p is

$$H(p) = \frac{1}{n}[k_1(p) + ... + k_n(p)],$$

and the Gauss-Kronecker curvature is

$$K(p) = k_1(p)k_2(p)...k_n(p).$$

Definition. Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. The point p on the hypersurface is called *absolutely umbilical* if all the principal curvatures satisfy $|k_1| = |k_2| = ... = |k_n|$. If all the points of a hypersurface are absolutely umbilical, then the hypersurface is called absolutely umbilical.

Definition. Define the curvature invariant

$$E(p) = (2)^{-\binom{n}{2}} \prod_{i \neq j} (|k_i| + |k_j|).$$

With this definition we prove the following.

Proposition 4. *Let $M^n \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and $k_1, k_2, \ldots, k_n$ be its principal curvatures in the ambient space $\mathbb{R}^{n+1}$ endowed with the canonical Euclidean metric. Let $p \in M$ be an arbitrary point. Denote by K the Gauss-Kronecker curvature. Then the following inequality holds true at every point $p \in M$:*

$$E(p) \geq |K(p)|^{\frac{n-1}{2}}.$$

Equality holds if p is an absolutely umbilical point.

Proof: Denote by $x_i = |k_i|$, for all $i = 1, 2, ..., n$. Then we have $\binom{n}{2}$ inequalities of the following form:

$$x_i + x_j \geq 2\sqrt{x_i x_j},$$

with equality if and only if $x_i = x_j$. By multiplying term by term these $\binom{n}{2}$ inequalities we obtain the stated result. □

It may be of interest to note that this inequality represents an extension of the inequality $H^2(p) \geq K(p)$ that holds true in the geometry of surfaces. To see this, let $n = 2$ in the statement above.

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