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Constructing a Triangle from Two Vertices and the Symmedian Point

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Abstract. Given three noncollinear points A, B, K, we propose a simple straightedge-and-compass construction of C such that K is the symmedian point of triangle ABC. Conditions for the construction to be possible and the number of solutions are studied in details.

1. Introduction

The problem of constructing a triangle ABC given two vertices A, B and one of the familiar centers (centroid, incenter, orthocenter, circumcenter), which appears in Wernick's famous list published in 1982 ([6, 7]), is easily solved. The case when the selected center is the symmedian point K being not listed, this article aims at offering a simple construction of the third vertex C from A, B, K and at studying the conditions on the three given points for the existence of a solution. In what follows, we always assume that the point K does not lie on the line AB.

2. The construction

The construction of a suitable point C will rest upon a not-very-well-known property of the symmedians, stated and proved in [1]. To keep this paper self-contained, it is presented here with a different proof.

Proposition 1. *Let ABC be a triangle and let B' be the reflection of B in A. The symmedian through A is the tangent at A to the circumcircle of $\Delta CAB'$.*

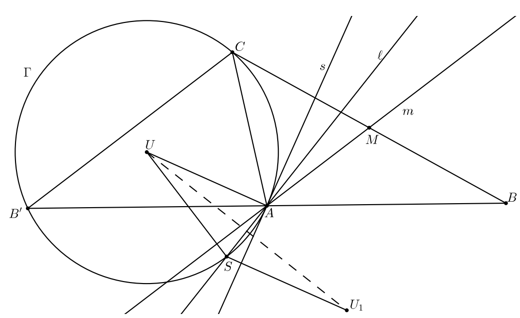

Figure 1

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Proof. Let M be the midpoint of BC and let s be the symmedian through A, that is, the reflection of the median $m = AM$ in the internal bisector ℓ of $\angle BAC$ (Figure 1). Let U be the center of the circle Γ through C, A, B' and S the point of intersection other than A of ℓ and Γ. Since ℓ is the external bisector of $\angle CAB'$, we have $SC = SB'$ and it follows that US is perpendicular to $B'C$, hence to $m = AM$. As a result, s is perpendicular to U_1S, where U_1 is the reflection of U in ℓ, hence to UA since UAU_1S is a rhombus. $\square$

Turning to our problem, consider the three points A, B, K and let B' be the reflection of B in A and A' be the reflection of A in B. Let γ_A (resp. γ_B) denote the unique circle passing through B' (resp. A') and tangent to AK at A (resp. tangent to BK at B). The point K is the symmedian point of ΔABC if and only if the lines AK and BK are the symmedians through A and B, respectively. Proposition 1 shows that this will occur if and only if the circles γ_A and γ_B both pass through C. Thus, the construction of the circles γ_A and γ_B readily yields the desired third vertex C as a common point of these circles, provided that this common point exists. Clearly, our problem has at most two solutions (see Figure 2, where two solutions C_1 and C_2 are obtained).

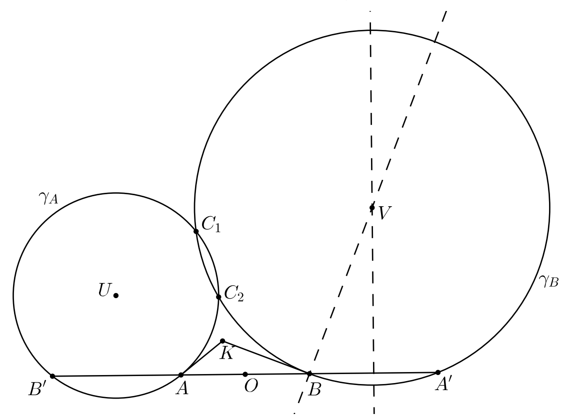

Figure 2

3. The discussion

From the construction above, the number of solutions for C depends on the relative position of the circles γ_A and γ_B. The way this position is related to the location of the points A, B, K is described in the following result:

Theorem 2. *The number of solutions for C is 0, 1, or 2, according as the sum $KA + KB$ is greater than, equal to, or less than $\frac{2AB}{\sqrt{3}}$.*

Proof. Let U and V be the centers of γ_A and γ_B, and let O denote the midpoint of AB (Figure 2). In a suitable system of axes with origin at O, we have $A(-\frac{c}{2}, 0),\ B(\frac{c}{2}, 0)$ (where $c = AB$) and we set $K(m, n)$. Expressing that V is the point of intersection of the perpendicular bisector of BA' (with equation $x = c$) and the perpendicular to BK at B (whose equation $\left(m - \frac{c}{2}\right) x + ny = \frac{c}{2}\left(m - \frac{c}{2}\right)$

is easily obtained), we find $V(c,v)$ where $v = \frac{c}{2n}\left(\frac{c}{2} - m\right)$. Similarly, $U(-c,u)$ where $u = \frac{c}{2n}\left(\frac{c}{2} + m\right)$ and a short calculation yields

$$UV^2 = \frac{c^2(m^2+4n^2)}{n^2}, \quad UA^2 = \frac{c^2((2m+c)^2+4n^2)}{16n^2}, \quad VB^2 = \frac{c^2((2m-c)^2+4n^2)}{16n^2}. \tag{1}$$

The condition $UV > |VB - UA|$ always holds because

$$UV^2 = 4c^2 + |v-u|^2 = 4c^2 + VB^2 - \frac{c^2}{4} + UA^2 - \frac{c^2}{4} - 2uv > VB^2 + UA^2 - 2UA \cdot VB.$$

Thus, the circles γ_A and γ_B are secant (resp. tangent) if and only if $UV < UA + VB$ (resp. $UV = UA + VB$).
Now, from (1), and with some algebra, we see that $UV < UA + VB$ is equivalent to

$$28n^2 + 4m^2 - c^2 < \sqrt{(4n^2+4m^2+c^2)^2 - 16c^2m^2},$$

which itself is equivalent to

$$4m^2 + 28n^2 < c^2 \quad \text{or} \quad 3m^2 + 12n^2 < c^2,$$

and finally to $3m^2 + 12n^2 < c^2$.
The latter condition means that K is interior to the ellipse $\mathcal{E}$ with foci A and B and major axis $\frac{2c}{\sqrt{3}}$. Clearly, the circles γ_A and γ_B are tangent if and only if K lies on $\mathcal{E}$. Since $\mathcal{E}$ is the set of all points P such that $PA + PB = \frac{2c}{\sqrt{3}}$, the proof is complete. □

4. Properties of the solutions

First, we examine the case when the problem has two solutions C_1 and C_2 (Figure 2) and prove the following

Proposition 3. *If $C_1 \neq C_2$ and K is the symmedian point of both ΔABC_1 and ΔABC_2, then the line C_1C_2 is a median of each triangle.*

Proof. We just observe that the midpoint O of AB has the same power with respect to γ_A and γ_B (since $OA \cdot OB' = \frac{3c^2}{4} = OB \cdot OA'$) and deduce that O is on the radical axis C_1C_2 of these two circles. □

In the case when the solution C is unique, the triangle ABC has a feature which is worth mentioning:

Proposition 4. *If K is the symmedian point of ΔABC for a unique point C, then $CA^2 + CB^2 = 2AB^2$.*

In other words, the triangle is a C-root-mean-square triangle (see [2, 4] for numerous properties of such triangles).

Proof. Again, O lies on the radical axis of γ_A and γ_B, hence on the common tangent to γ_A and γ_B at C. It follows that $\frac{3c^2}{4} = OC^2$; but, CO being the length of the median from C, we also have $4CO^2 = 2(CA^2 + CB^2) - c^2$ and a short calculation gives the desired relation. □

Note in passing that when K traverses the ellipse $\mathcal{E}$, the corresponding unique solution C traverses the circle with center O and radius $\frac{c\sqrt{3}}{2}$ (less the common points with the line AB, of course).

5. Further remarks

To conclude, we draw some interesting corollaries of the results above.
In an arbitrary triangle ABC, not only does the symmedian point K satisfy $KA + KB \leq \frac{2AB}{\sqrt{3}}$, but it also satisfies the similar inequalities associated with the pairs of vertices (B, C) and (C, A):

Corollary 5. *The symmedian point K of any triangle ABC is located in the common part of the interiors of the three ellipses with foci A and B, with foci B and C, with foci C and A, and with respective major axes $\frac{2AB}{\sqrt{3}}$, $\frac{2BC}{\sqrt{3}}$, $\frac{2CA}{\sqrt{3}}$ (boundary included).*

It is known that K is always interior to the circle with diameter GH where G is the centroid and H the orthocenter (see [3]). However, the common part interior to the three ellipses can be much smaller (Figure 3).

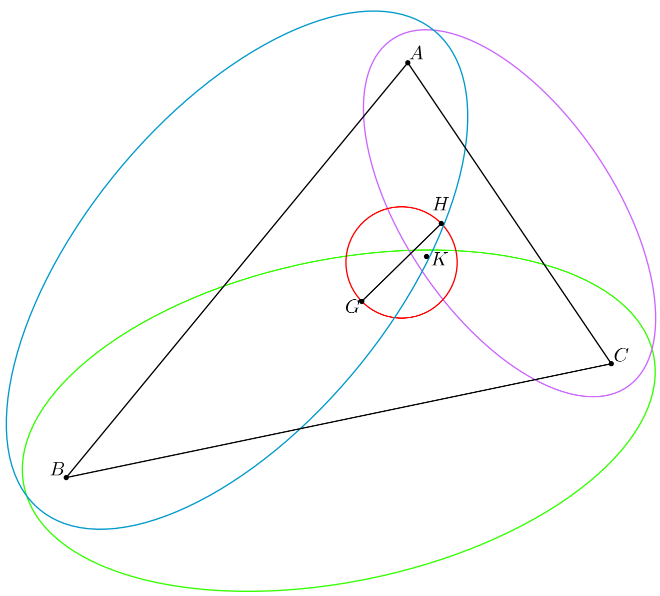

Figure 3

Lastly, we deduce a nice geometric inequality, perhaps difficult to prove directly:

Corollary 6. *Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC with sides $a = BC, b = CA, c = AB$. Then, the following inequality holds:*

$$\max\{bm_c + cm_b, cm_a + am_c, am_b + bm_a\} \leq \frac{a^2 + b^2 + c^2}{\sqrt{3}}. \tag{2}$$

Interestingly, substituting m_a, m_b, m_c, $\frac{3a}{4}$, $\frac{3b}{4}$, $\frac{3c}{4}$ for a, b, c, m_a, m_b, m_c, respectively, leaves the inequality unchanged, meaning that the inequality is its own median-dual ([5] p. 109). No hope then to derive it from a known inequality through median duality!

Proof. We know that the symmedian point K of ΔABC is the center of masses of (A, a^2), (B, b^2), (C, c^2), that is, $(a^2+b^2+c^2)K = a^2A + b^2B + c^2C$. It follows that $(a^2+b^2+c^2)\overrightarrow{AK} = b^2\overrightarrow{AB} + c^2\overrightarrow{AC}$ and so

$$\begin{aligned}(a^2+b^2+c^2)^2AK^2 &= b^4c^2 + c^4b^2 + (b^2c^2)(2\overrightarrow{AB}\cdot\overrightarrow{AC})\\ &= b^2c^2(b^2+c^2+b^2+c^2-a^2)\\ &= 4b^2c^2m_a^2.\end{aligned}$$

As a result, $KA = \frac{2bcm_a}{a^2+b^2+c^2}$ and similarly we obtain $KB = \frac{2cam_b}{a^2+b^2+c^2}$. The inequality $KA + KB \leq \frac{2c}{\sqrt{3}}$ now rewrites as $bm_a + am_b \leq \frac{a^2+b^2+c^2}{\sqrt{3}}$. Cyclically, the numbers $bm_c + cm_b$ and $cm_a + am_c$ are also less than or equal to $\frac{a^2+b^2+c^2}{\sqrt{3}}$. □

Note that equality holds if and only if ΔABC is a root-mean-square triangle, that is, if and only if one of the relations $2a^2 = b^2 + c^2,\ 2b^2 = c^2 + a^2,\ 2c^2 = a^2 + b^2$ is satisfied.

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