Forum Geometricorum
Volume 18 (2018) 135–140.

FORUM GEOM
ISSN 1534-1178

The "Circle" of Apollonius in Hyperbolic Geometry

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Abstract. In Euclidean geometry the circle of Apollonius is the locus of points in the plane from which two collinear adjacent segments are perceived as having the same length. In Hyperbolic geometry, the analog of this locus is an algebraic curve of degree four which can be bounded or "unbounded". We study this locus and give a simple description of this curve using the Poincaré half-plane model. In the end, we give the motivation of our investigation and calculate the probability that three collinear adjacent segments can be seen as of the same positive length under some natural assumptions about the setting of the randomness considered.

1. Introduction

There are at least four circles known as Apollonius circles in the history of classical geometry. We are only concerned with only one of them: "the set of all points whose distances from two fixed points are in a constant ratio ρ ($\rho \neq 1$) " (see [1], [5]). In Euclidean geometry this is equivalent to asking for the locus of points P satisfying $\angle APB \equiv \angle BPC$, given fixed collinear points A, B and C. This same locus is the focus of our investigation in Hyperbolic geometry.

We are going to use the half-plane model, $\mathbb{H}$ (viewed in polar coordinates as the upper-half, i.e., $\theta \in (0,\pi)$), to formulate our answer to this question (see Anderson [2] for the terminology and notation used). Without loss of generality we may assume that the three points are on the y-axis: $A(0,a)$, $B(0,b)$ and $C(0,c)$, with real positive numbers a, b and c such that $a > b > c$.

Theorem 1. *Given points A, B and C as above, the set of points $P(x,y)$ in the half-plane $\mathbb{H}$, characterized by the equality $\angle APB \equiv \angle BPC$ in $\mathbb{H}$ is the curve given in polar coordinates by*

$$r^4(2b^2-a^2-c^2) = 2r^2(a^2c^2-b^4)\cos(2\theta) + b^2(2a^2c^2-a^2b^2-c^2b^2). \quad (1)$$

Moreover,

(i) if $b = \left(\frac{a^2+c^2}{2}\right)^{\frac{1}{2}}$, this curve is half of the hyperbola of equation

$$r^2\cos(2\theta) + b^2 = 0, \;\; \theta \in (\frac{\pi}{4}, \frac{3\pi}{4}),$$

Publication Date: May 17, 2018. Communicating Editor: Paul Yiu.

(ii) if $b=\sqrt{ac}$ this curve is the semi-circle $r=b$, $\theta\in(0,\pi)$,
(iii) if $b=\left(\frac{a^{-2}+c^{-2}}{2}\right)^{-\frac{1}{2}}$, this curve is half of the lemniscate of equation

$$r^2+b^2\cos(2\theta)=0,\ \ \theta\in(\frac{\pi}{4},\frac{3\pi}{4}).$$

In most of the textbooks the Circle of Apollonius is discussed in conjunction with the Angle Bisector Theorem: *"The angle bisector in a triangle divides the opposite side into a ratio equal to the ratio of the adjacent sides."* Once one realizes that the statement can be equally applied to the exterior angle bisector, then the Circle of Apollonius appears naturally (Figure 1), since the two angle bisectors are perpendicular.

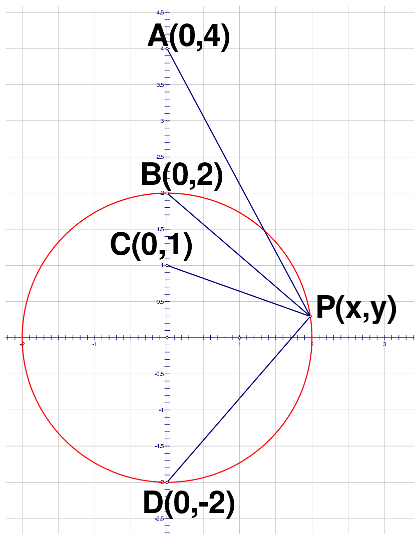

Figure 1. The Circle $x^2+y^2=4$

For instance, an easy exercise in algebra shows that the circle of equation $x^2+y^2=4$ is equivalent to

$$\frac{\sqrt{x^2+(y-4)^2}}{\sqrt{x^2+(y-1)^2}}=\frac{BA}{BC}=\frac{DA}{DC}=2,$$

taking $A(0,4)$, $B(0,2)$, $C(0,1)$ and $D(0,-2)$. Similar calculations can be employed to treat the general situation, i.e., taking $A(0,a)$, $B(0,b)$ and $C(0,c)$ with real positive numbers a, b and c such that $a>b>c$. Then we can state the well-known result:

Theorem 2 (Apollonius)**.** *Given points A, B and C as above, the set of points $P(x,y)$ in the plane characterized by the equality $\angle APB\equiv\angle BPC$ is*
(i) the line of equation $y=(a+c)/2$ if $b=(a+c)/2$;
(ii) the circle of equation $x^2+y^2=b^2$, if $b<(a+c)/2$ and $b^2=ac$.

Let us observe that the statement of this theorem does not reduce the generality since the y-coordinate of point D can be shown to be $y_D = (2ac - bc - ab)/(a + c - 2b)$ and one can take the origin of coordinates to be the midpoint of $\overline{BD}$. This turns out to happen precisely when $b^2 = ac$.

It is interesting that each of the special cases in Theorem 1, can be accomplished using integer values of a, b and c. This is not surprising for the Diophantine equation $b^2 = ac$ since one can play with the prime decomposition of a and c to get ac a perfect square. For the equation $2b^2 = a^2 + c^2$ one can take a Pythagorean triple and set $a = |m^2 + 2mn - n^2|$, $c = |m^2 - 2mn - n^2|$ and $b = m^2 + n^2$ for $m, n \in \mathbb{N}$. Perhaps it is quite intriguing for some readers that the last Diophantine equation $2a^2c^2 = a^2b^2 + c^2b^2$ is satisfied by the product of some quadratic forms, namely

$$\begin{aligned} a &= (46m^2 + 24mn + n^2)(74m^2 + 10mn + n^2), \\ b &= (46m^2 + 24mn + n^2)(94m^2 + 4mn - n^2), \\ \text{and } \ c &= (94m^2 + 4mn - n^2)(74m^2 + 10mn + n^2), \ \text{ for } \ m, n \in \mathbb{Z}. \end{aligned}$$

Another surprising fact is that the locus is a circle both in Theorem 1 and Theorem 2 if $b^2 = ac$.

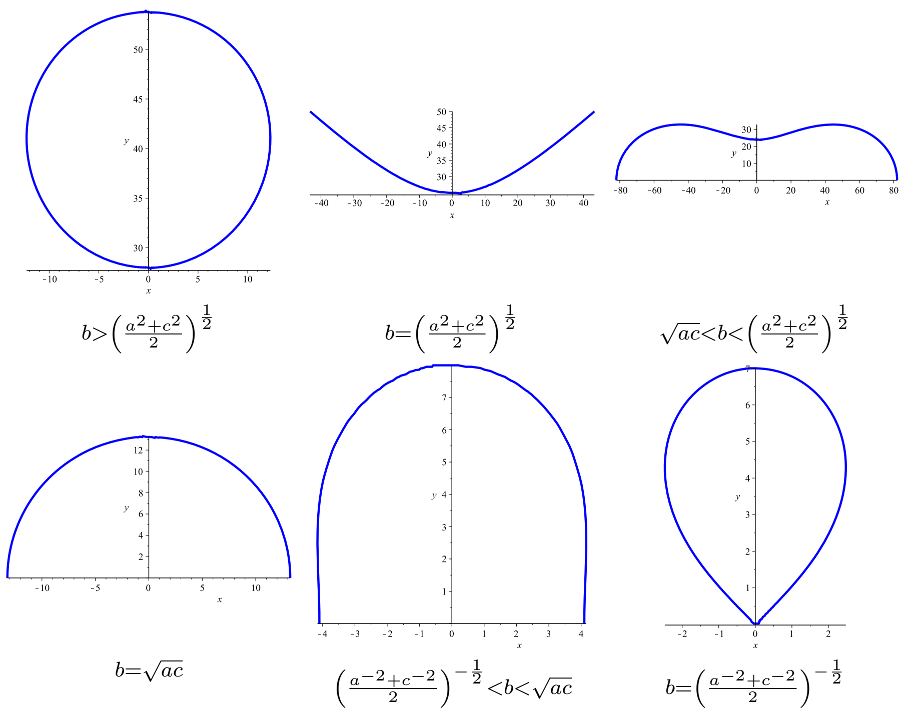

Figure 2. The curve in polar coordinates (a=35, c=5), in $\mathbb{H}$

In Figure 2, we included all of the possible shapes of the locus in Theorem 1 except for the case $b < \left(\frac{a^{-2}+c^{-2}}{2}\right)^{-\frac{1}{2}}$ which is similar to the case $b > \left(\frac{a^2+c^2}{2}\right)^{\frac{1}{2}}$. We notice a certain symmetry of these cases showing that the hyperbola ($b =$

$\left(\frac{a^2+c^2}{2}\right)^{\frac{1}{2}}$) is nothing else but a lemniscate in hyperbolic geometry. With this identification, it seems like the curves we get, resemble all possible shapes of the intersection of a plane with a torus.

In the next section we will prove Theorem 1.

2. Proof of Theorem 1

Let us consider a point P of coordinates (x, y) with the given property as in Figure 3 which is not on the line $\overline{AC}$, ($x \neq 0$). Then the Hyperbolic lines determined by P and the three points A, B and C are circles orthogonal on the x-axis. We denote their centers by $A'(a', 0)$, $B'(b', 0)$ and $C'(c', 0)$. The point A' can be

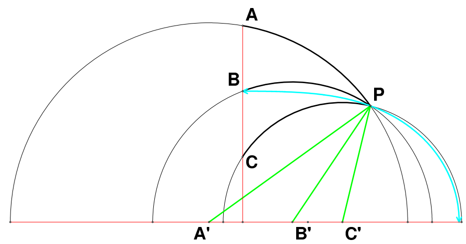

Figure 3. The point P and the lines determined by it with A, B and C

obtained as the intersection of the perpendicular bisector of $\overline{PA}$ and the x-axis. Similarly we obtain the other two points B' and C'. The equation (Y as a function of X) of the perpendicular bisector of $\overline{PA}$ is $Y - \frac{y+a}{2} = -\frac{x}{y-a}(X - \frac{x}{2})$ and so $a' = \frac{x^2+y^2-a^2}{2x}$. Similar expressions are then obtained for b' and c', i.e., $b' = \frac{x^2+y^2-b^2}{2x}$ and $c' = \frac{x^2+y^2-c^2}{2x}$. This shows that the order of the points A', B' and C' is reversed ($a' < b' < c'$). The angle between the Hyperbolic lines $\overleftrightarrow{PA}$ and $\overleftrightarrow{PA}$ is defined by the angle between the tangent lines to the two circles at P, which is clearly equal to the angle between the radii corresponding to P in each of the two circles. So, $m_{\mathbb{H}}(\angle APB) = m(\angle A'PB')$ and $m_{\mathbb{H}}(\angle BPC) = m(\angle B'PC')$. This equality is characterized by the proportionality given by the Angle Bisector Theorem in the triangle $PA'C'$:

$$\frac{PA'}{PC'} = \frac{A'B'}{B'C'} \Leftrightarrow \frac{\sqrt{(x^2 - y^2 + a^2)^2 + 4x^2y^2}}{\sqrt{(x^2 - y^2 + c^2)^2 + 4x^2y^2}} = \frac{a^2 - b^2}{b^2 - c^2}.$$

Using polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, we observe that $x^2 - y^2 = r^2\cos 2\theta$ and $2xy = r^2\sin 2\theta$. Hence the above equality is equivalent to

$$(r^4 + 2a^2r^2\cos 2\theta + a^4)(b^2 - c^2)^2 = (r^4 + 2c^2r^2\cos 2\theta + c^4)(a^2 - b^2)^2.$$

One can check that a factor of $(a^2 - c^2)$ can be simplified out and in the end we obtain (1).

3. Four points “equally” spaced and our motivation

Our interest in this locus was motivated by the Problem 11915 in this Monthly ([4]). This problem stated: *Given four (distinct) points A, B, C and D in (this) order on a line in Euclidean space, under what conditions will there be a point P off the line such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure? (Figure 4)*

It is not difficult to show, using two Apollonius circles, that the existence of such a point P is characterized by the inequality involving the cross-ratio

$$[A, B; C, D] = \tfrac{BC}{BA}/\tfrac{DC}{DA} < 3. \tag{2}$$

We were interested in finding a similar description for the same question in Hyperbolic space. It is difficult to use the same idea of the locus that replaces the Apollonius circle in Euclidean geometry due to the complicated description as in Theorem 1. Fortunately, we can use the calculation done in the proof of Theorem 1 and formulate a possible answer in the new setting. Given four points A, B, C and D in (this) order on a line in the Hyperbolic space, we can use an isometry to transform them on the line $x = 0$ and having coordinates $A(0, a)$, $B(0, b)$, $C(0, c)$ and $D(0, d)$ with $a > b > c > d$. The the existence of a point P off the line $x = 0$, such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure in the Hyperbolic space is equivalent to the existence of P in Euclidean space corresponding to the points A', B', C' and D' as constructed in the proof of Theorem 1. Therefore, the answer to the equivalent question posed in the Problem 11915 but in Hyperbolic geometry, is in terms of a similar inequality

$$[A', B'; C', D'] = \tfrac{B'C'}{B'A'}/\tfrac{D'C'}{D'A'} < 3 \Leftrightarrow \frac{(b^2 - c^2)(a^2 - d^2)}{(a^2 - b^2)(c^2 - d^2)} < 3. \tag{3}$$

Now we can use (2) and (3) to compute the following natural corresponding geometric probability: *if two points B and C are randomly selected (uniform distribution with respect to the arc-length in hyperbolic geometry) on the segment $\overline{AD}$ (B being the closest to A), what is the probability that a point P off the line $\overline{AD}$ exists, such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure? (Figure 4)*

In Euclidean geometry this turns out to be equal to

$$P_e = \frac{15 - 16 \ln 2}{9} \approx 0.4345$$

The inequality (3) gives us the similar probability in the Hyperbolic space:

$$P_h = \frac{2\sqrt{5} \ln(2 + \sqrt{5}) - 5}{5 \ln 2} \approx 0.4201514924$$

where the uniform distribution here means that it is calculated with respect to the measure $\frac{1}{y}dy$ along the y-axis.

Our probability question is even more natural in the setting of spherical geometry. Due to the infinite nature of both Euclidean and Hyperbolic spaces the

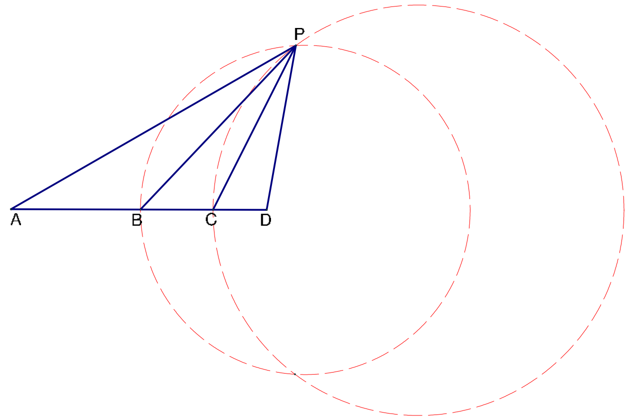

Figure 4. The point P and the lines determined by it with A, B , C and D

geometric probability question makes sense only in limiting situations. In the case of spherical geometry we can simply ask (and leave it as an open question):

Given a line in spherical geometry and four points on it, chosen at random with uniform distribution, what is the probability that the points look equidistant from a point on the sphere that is not on that line?

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