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Rectangles Circumscribing a Quadrangle

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Abstract. In this article we study the circumscribed rectangles about a quadrangle and in particular their extrema with respect to the area. We show that there are two such extrema represented by two similar rectangles. In addition we study the four similarities interchanging these extremal rectangles and show the existence of two twin quadrangles with given extremal rectangles.

1. Circumscribing rectangles

In this article we study the configuration created by an arbitrary non-orthodiagonal quadrangle $ABCD$ and the circles $\{\alpha, \beta, \gamma, \delta\}$ on diameters the sides of the quadrangle (See Figure 1). These circles carry the vertices of all the rectangles

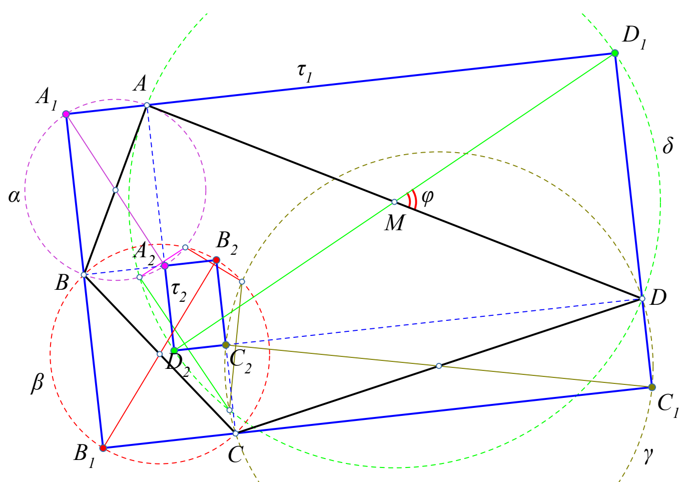

Figure 1. Rectangles circumscribing the quadrangle $ABCD$

that circumscribe the quadrangle, and the figure shows also two prominent such rectangles $\{\tau_1, \tau_2\}$, which are similar to each other. They represent two extrema of the signed area function $f(\phi)$ of the circumscribing rectangles. The big one τ_1 can be considered to have positive area and is the greatest, w.r. to the area, rectangle circumscribing $ABCD$. The small one τ_2 corresponds then to the minimal extremal value of the signed area function, which in section 3 is proved to be a periodic sinuisoidal function with two extrema, as seen in the corresponding graph in figure 2. In the graph are also noticed the two extrema, which occur at points $\{\phi = 0, \phi = \pi\}$, corresponding to diametral points on each one of the circles $\{\alpha, \beta, \gamma, \delta\}$. I will refer in the sequel to the two prominent rectangles as the

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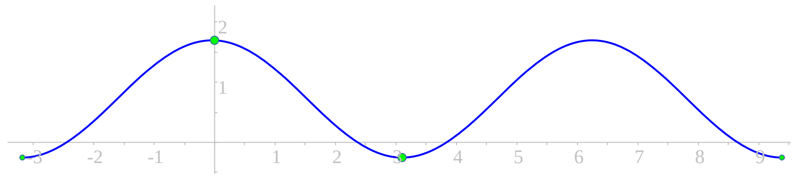

Figure 2. Graph of the area function

extremal rectangles of $ABCD$, distinguishing also between the *maximal* τ_1 with area E_b and the *minimal* one τ_2 with area E_s. The restriction on non-orthodiagonal quadrangles, i.e. those that have their diagonals not orthogonal, is equivalent with the condition, that not three of the circles $\{\alpha, \beta, \gamma, \delta\}$ are concurrent at a point, which in turn will be seen below to be equivalent with the non-degeneration of the minimal rectangle τ_2 or, equivalently, the condition $E_s \neq 0$.

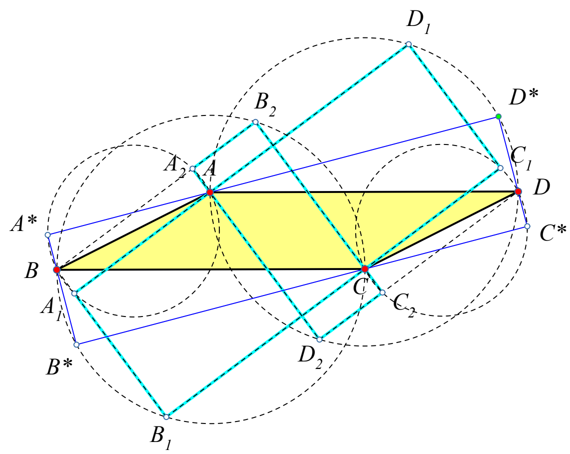

Figure 3. Circumscribing but not enclosing

At this point it should be stressed the difference of *circumscribing* to *enclosing*. The rectangles we deal with *circumscribe*, i.e. have their side-lines passing through the vertices of the quadrangle of reference $ABCD$, but may not *enclose* it in their inner domain. This is illustrated by figure 3, in the case in which the quadrangle of reference $ABCD$ is a parallelogram. Neither of the two extremal rectangles $\{\tau_1 = A_1B_1C_1D_1, \tau_2 = A_2B_2C_2D_2\}$ encloses $ABCD$, as it does the rectangle $A^*B^*C^*D^*$. The set of enclosing rectangles is a subset of the circumscribing ones. Thus, the maximal enclosing may coincide with the maximal circumscribing, but it can also be different from this, having E_b as an upper bound for its area. The problem of maximal *enclosing* rectangles has more interesting computational aspects, as can be seen e.g. in [10], where it is handled for the special case of rectangles enclosing paralellograms. Instead, the configuration of rectangles *circumscribing* a quadrangle is connected with interesting geometric structures, as will be hopefully seen in the following sections.

2. Four similar kites

In the case of non-orthodiagonal quadrangles $q = ABCD$, the other than the vertices of q intersection points of the circles $\{\alpha \cap \beta, \beta \cap \gamma, \gamma \cap \delta, \delta \cap \alpha\}$ can be easily identified with the projections of the vertices of q on its diagonals. These points define a similar to q quadrangle $q' = A'B'C'D'$ (See Figure 4).

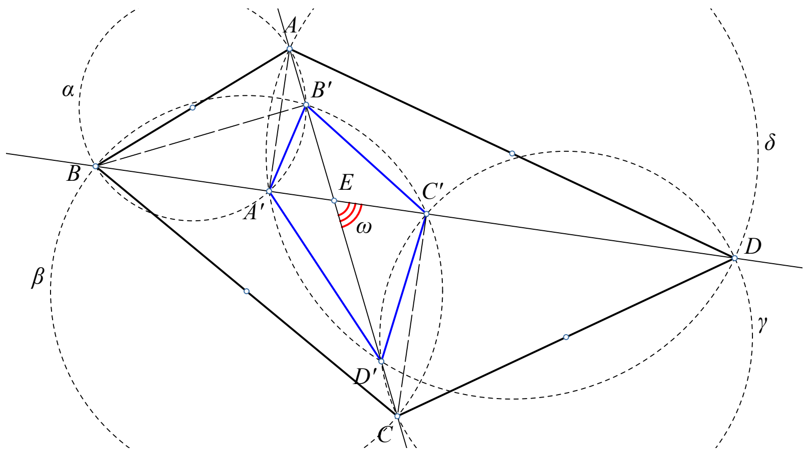

Figure 4. Projecting the vertices on the diagonals

Lemma 1. *The quadrangle $q' = A'B'C'D'$ is similar to q and inversely oriented to it. The similarity ratio of q' to q is equal to $\cos(\omega)$, where $0 < \omega < \pi/2$ is the angle of the diagonals of q.*

Proof. By the figure, which shows that triangles $\{ABC, A'B'C'\}$ are similar. In fact, $\widehat{C'A'B'} = \widehat{CAB}$ by the cyclic quadrangle $AB'A'B$, and $\widehat{A'C'B'} = \widehat{ACB}$ by the cyclic quadrangle $BB'C'C$. Analogously is seen that $\{BCD, B'C'D'\}$ are similar. The similarity ratio is the ratio of the diagonals $A'C'/AC = \cos(\omega)$. The reversing of the orientation results from the fact that the two quadrangles have the same lines as diagonals, but their roles are interchanged, the diagonal carrying $\{B, D\}$ now carrying $\{A', C'\}$, etc. □

This quadrangle is of significance for the location of the extremal circumscribing rectangles, since, as will be seen below, the vertices of these rectangles lie on the medial lines of the sides of q'. In fact, each side of q', together with its medial line, which defines a diameter of the corresponding circle, creates a *kite* inscribed in the corresponding circle. Figure 5 shows the kite $D_1A'D_2D'$, inscribed in the circle δ. It is created from the diameter D_1D_2 of the circle δ, which is orthogonal to its chord $A'D'$, which is a side of q'. The chord is non degenerate $(A' \neq D')$, precisely under the general assumption made, that the quadrangle is non orthodiagonal, equivalently, that the three circles $\{\alpha, \gamma, \delta\}$ are not concurrent at a point. There are then three other similarly defined kites, corresponding to the other sides of q'. This similarity is an instance of a more general one concerning the four

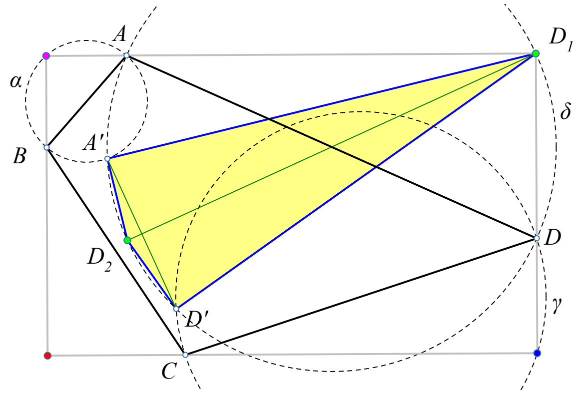

Figure 5. Kites carrying the vertices of the extremal rectangles

cyclic quadrangles defined by any circumscribed rectangle $\sigma_1 = A_1B_1C_1D_1$ and its *antipodal* $\sigma_2 = A_2B_2C_2D_2$, created by the diametral points of the vertices of σ_1, on the respective circles $\{\alpha, \beta, \gamma, \delta\}$ (See Figure 6). Next lemma lists some

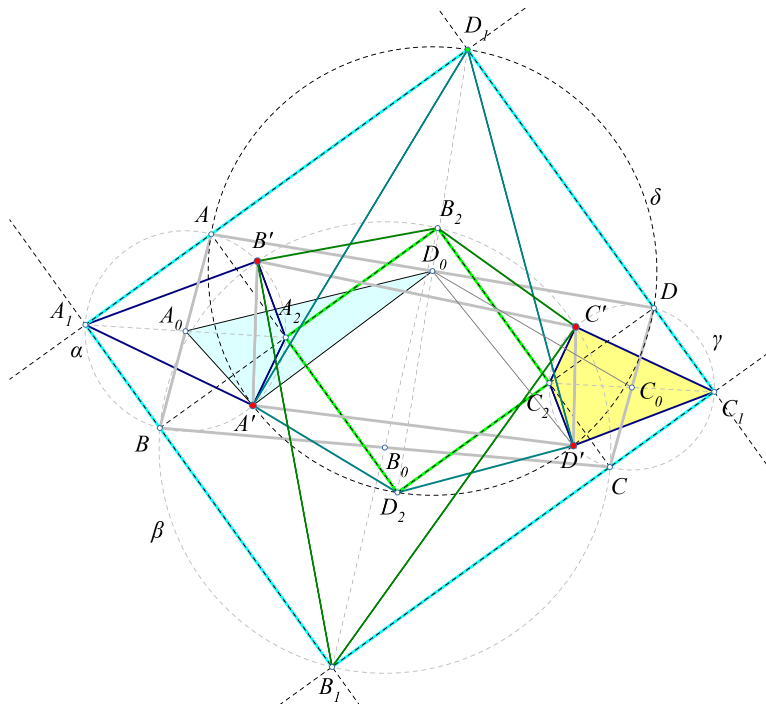

Figure 6. Circumscribed $\sigma_1 = A_1B_1C_1D_1$, “antipodal” $\sigma_2 = A_2B_2C_2D_2$

fundamental properties of this figure.

Lemma 2. *With the notation and the conventions introduced in this section, and denoting by* $\{A_0, B_0, C_0, D_0\}$ *the middles of the sides* $\{AB, BC, CD, DA\}$*, the following are valid properties.*

(1) *The circumscribing rectangles* $\{\sigma_1, \sigma_2\}$ *have parallel sides.*
(2) *The triangles* $\{A'D_1A_1, A'D_2A_2, A'D_0A_0\}$ *are similar, the same property holding for the cyclic permutations of the letters* $\{A, B, C, D\}$.
(3) *The cyclic quadrangles* $\{D_1A'D_2D', A_1A'A_2B'\}$ *are similar, the same property holding for the cyclic permutations of the letters* $\{A, B, C, D\}$.
(4) *The quadrangles of nr-3 have two right angles and the other two are equal to* $\pi/2 \pm \omega$*, where* $\omega \leq \pi/2$ *is the angle of the diagonals of* $ABCD$.

Proof. Nr-1 results from the right angles $\{\widehat{D_2AD_1}, \widehat{A_2AA_1}\}$. This implies that $\{A, A_2, D_2\}$ are collinear and their line is parallel to A_1B_1. Analogously is seen the parallelity of the other pairs of sides.

Nr-2 is a standard exercise in elementary euclidean geometry ([7, p.290]). The map $D_1 \mapsto A_1$ of the circle δ onto circle α can be described by a similarity $A_1 = f_A(D_1)$ with center, or *invariant point* ([5, p.72]) at A angle $\phi_A = \widehat{D_0A'A_0} = \widehat{BAD}$ and ratio $k_A = A'A_0/A'D_0$.

Nr-3 results by showing that the similarity f_A of *nr-2* maps one quadrangle onto the other $f_A(D_1A'D_2D') = A_1A'B'A_1$, which follows by a simple angle chasing argument.

Nr-4 follows from a simple angle chasing argument, since $\widehat{A'D_1D'} = \widehat{A'D_0D'}/2$, which is $\pi - 2(\phi + \psi)$, where $\phi = \widehat{AD_0A_0}$ and $\psi = \widehat{C_0D_0D}$. The claim follows from the fact that $\{A_0D_0, D_0C_0\}$ are parallel to the diagonals. □

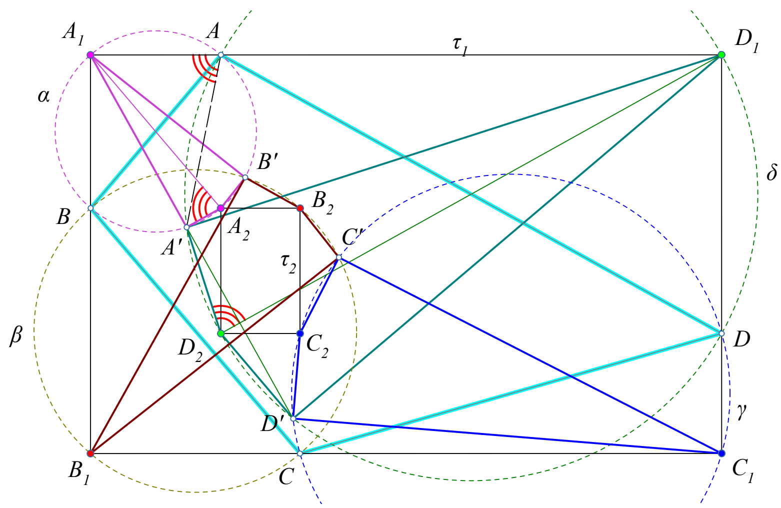

Figure 7. The four similar kites of the quadrangle $ABCD$

Figure 7 shows the four similar kites, resulting from the previous lemma in the case the diagonals of the cyclic quadrangles like $A_1A'A_2B'$ are orthogonal. In section 4 we will show that the rectangles $\{\tau_1 = A_1B_1C_1D_1, \tau_2 = A_2B_2C_2D_2\}$ of this figure are the extremal circumscribed rectangles of q.

Lemma 3. *In the special case in which $\{D_1, D_2\}$ are vertices of the kite $D_1A'D_2D'$, the corresponding circumscribing rectangles $\{\tau_1, \tau_2\}$ are similar, the similarity ratio, being equal to the ratio $D'D_2/D'D_1$ of the sides of the kite.*

Proof. The similarity of the rectangles for the claimed particular position of $D_1 \in \delta$ is implied from the similarity of the kites. In fact, for that position of D_1 on δ, the triangles $\{C'B_2C_2, C'B_1C_1\}$ are similar and their similarity ratio is equal to the ratio $C'B_2/C'B_1$ of the sides of a kite, which is the same for all four kites. This shows that $B_2C_2/B_1C_1 = C'B_2/C'B_1$ and an analogous argument shows that the last ratio is also equal to C_2D_2/C_1D_1. □

These preliminary remarks, made in this and the previous section, show the existence and the way to construct the two similar extremal rectangles circumscribing the quadrangle $ABCD$. It remains to justify their names and show actually their extremal property. This will be done in section 4, after a short study of the function of the signed area of the circumscribing rectangle.

3. The area function

In order to study the signed area function of the circumscribed rectangles of the quadrangle $q = ABCD$, it suffices to consider their half, defined by a diagonal of them. Figure 8 shows the right angled triangle XYZ, which is such a half of a

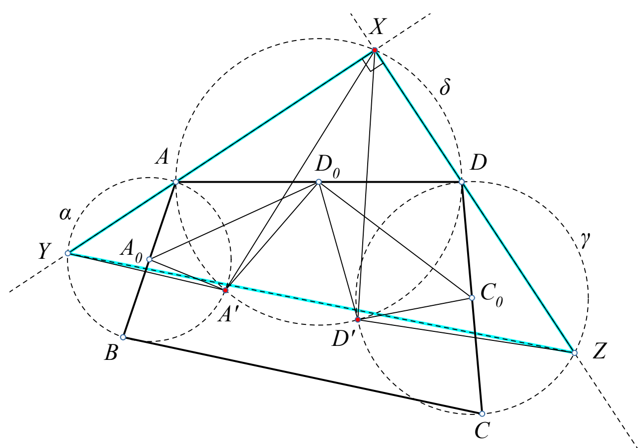

Figure 8. A half of the rectangle circumscribing $ABCD$

circumscribing rectangle. The study of its area can be done by using the naturally defined similarity transformations introduced in the previous section. In fact, the correspondence $X \mapsto Y$, for X varying on the circle δ, can be expressed by the similarity $Y = f_A(X)$ with center at A', angle $\phi_A = \widehat{D_0A'A_0} = \widehat{A}$ and ratio $k_A = A'A_0/A'D_0 = AB/AD$, points $\{D_0, A_0, C_0\}$ being the middles respectively of the sides $\{AD, AB, CD\}$. This similarity can be conveniently described using complex numbers, in the form

$$Y = A' + k_A \cdot e^{i\phi_A}(X - A') \qquad \Rightarrow \qquad Y - X = (k_A \cdot e^{i\phi_A} - 1) \cdot (X - A').$$

Analogously, the correspondence $X \mapsto Z$ can be described by the similarity f_D centerred at D', in the form

$$Z = D' + k_D \cdot e^{i\phi_D}(X - D') \qquad \Rightarrow \qquad Z - X = (k_D \cdot e^{i\phi_D} - 1) \cdot (X - D'),$$

where $\phi_D = \widehat{D_0 D' C_0} = \widehat{D}$ and $k_D = D'C_0/D'D_0 = DC/AD$. Using further complex numbers ([4, p.70], [8, p.48]), the signed area of the triangle $\sigma = XYZ$ can be expressed by the formula

$$\begin{aligned} [XYZ] &= \frac{1}{2} Im((\overline{Y - X})(Z - X)) \\ &= \frac{1}{2} Im((\overline{(k_A e^{i\phi_A} - 1)(X - A')})(k_D e^{i\phi_D} - 1)(X - D')) \\ &= \frac{1}{2} Im((k_A e^{-i\phi_A} - 1)(k_D e^{i\phi_D} - 1)\overline{(X - A')}(X - D')). \end{aligned}$$

Identifying D_0 with the origin and $\{A, D\}$ on the x-axis and symmetric w.r.t. to D_0, we arive for the signed area at an expression of the form

$$[XYZ] = N - Im(M(\overline{A'}X + D'\overline{X})), \tag{1}$$

where, since $\overline{X}X = |D_0 A|^2$, the numbers $\{M, N\}$ are respectively the complex and real constants

$$\begin{aligned} M &= \frac{1}{2}(k_A e^{-i\phi_A} - 1)(k_D e^{i\phi_D} - 1), \\ N &= Im(M(\overline{X}X + \overline{A'}D')) = Im(M(|D_0 A|^2 + \overline{A'}D')). \end{aligned}$$

Setting X in polar form, $X = |D_0 A| e^{i\psi}$, in formula (1), we see, after a short calculation, that the signed area $[XYZ]$ is a sinuisoidal periodic function of the polar angle ψ, as this was suggested by figure 2. A second conclusion from this formula results by replacing X with $X' = -X = |D_0 A| e^{i(\psi + \pi)}$, which corresponds to the construction of the circumscribed rectangle, starting this time with the diametral point X' of X w.r. to the circle δ. The result is

$$[XYZ] + [X'Y'Z'] = 2N, \tag{2}$$

which shows that the signed areas corresponding to diametral points sum to a constant. Regarding the value of the real constant N, it can be determined by considering particular positions of the circumscribed rectangles. These are positions $\{D_{AC}, D_{AB}\}$ of D_1 (See Figure 9), for which the sides of $\sigma_1 = A_1 B_1 C_1 D_1$ become parallel and not identical to a diagonal of $ABCD$. Then it is trivial to see that $\sigma_2 = A_2 B_2 C_2 D_2$ degenerates and the area of the σ_1 becomes, up to sign, equal to $|AC||BD|\sin(\omega) = 2|ABCD|$, where $\omega < \pi/2$ the angle of the diagonals. Selecting the orientation of the triangle ABD and of one such rectangle to be positive, we see that the area of the rectangle σ_1 varying continiusly in dependence of $D_1 \in \delta$, is positive for D_1 varying on the greater arc of δ, defined by its chord $A'D'$, and negative for D_1 on the small arc defined by that chord. As a consequence, the maximal and minimal circumscribing rectangles $\{\tau_1, \tau_2\}$

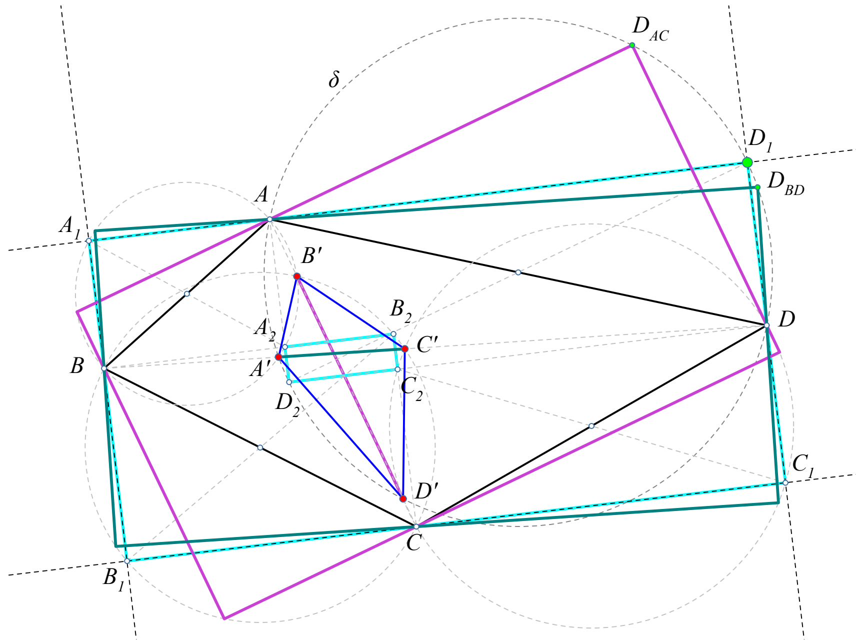

Figure 9. Positions for which $\sigma_2 = A_2B_2C_2D_2$ degenerates

have opposite orientations, their areas $\{E_b, E_s\}$ have opposite signs and we can set $N = |ABCD|/2 > 0$, which leads to the formula

$$E_b + E_s = 2|ABCD|. \tag{3}$$

There is an interesting configuration, which should be considered here, concerning the case in which the aforementioned arcs of δ are equal. This is the case of self-intersecting quadrangles, whose signed area $|ABCD| = 0$. Since the

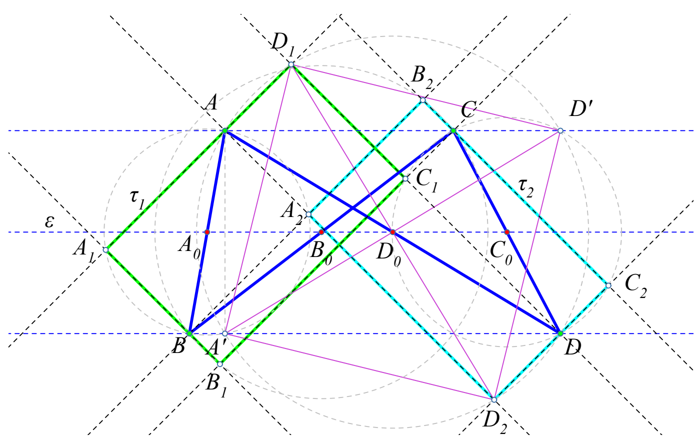

Figure 10. Side-middles on a line ε

area of a quadrangle $q = ABCD$ is a multiple of the area of the corresponding

Varignon parallelogram $q_0 = A_0B_0C_0D_0$, with vertices the middles of the sides of $ABCD$, this condition is equivalent with the degeneration of q_0 or, equivalently, the collinearity of these middles. Figure 10 shows such a case, together with the two inversely oriented extremal rectangles, which, as is easily seen, now are congruent. The corresponding kites, like $D_1A'D'D_2$, are squares, the diagonals $\{AC, BD\}$ are parallel and their angle can be considered to be $\omega = 0$. In this case the quadrangle $q' = A'B'C'D'$, of the projections of the vertices of q on its diagonals, is the reflected of q w.r. to the line ε carrying the middles, hence congruent to it. Finally, it can be easily verified that the ratio of the sides of the extremal rectangles D_1C_1/D_1A_1 is equal to the ratio of the diagonals AC/BD, which is something shown below (corollary 9) to be generally valid for all quadrangles.

We summarize the results so far in the following theorem.

Theorem 4. *Under the notation and conventions of this section the following are valid properties.*

(1) *The signed area function of the circumscribed rectangle is a sinuisoidal periodic function of the polar angle ψ of a side of the rectangle.*
(2) *The sum of the signed areas of a circumscribed rectangle and its diametral, obtained for $\psi + \pi$, is constant.*
(3) *The areas of the two extremal rectangles have opposite signs and their sum is $E_b + E_s = 2|ABCD|$.*
(4) *The two extremal rectangles are congruent precisely when $|ABCD| = 0$, equivalently, when the middles of the sides of $ABCD$ are collinear.*

4. The extremal rectangles

The question of the extremal circumscribed rectangles can be settled using further their half, defined by a diagonal of them, as in the preceding section, on the ground of figure 8. To this figure refers also next theorem, establishing the connection of the extremals with the kites introduced in section 2 for non-orthodiagonal quadrangles.

Theorem 5. *The area of the triangle XYZ obtains an extremal value, precisely when the triangle $A'XD'$ is isosceles, having $|XA'| = |XD'|$.*

Proof. Since by theorem 4 the extremals have non zero areas, in the task to locate them, we can use absolute values. Using then the expressions for the sides of XYZ of the preceding section, we see that the product of these sides is

$$\begin{aligned} |X-Y||X-Z| &= |(k_A e^{i\phi_A} - 1)(X - A')||(k_D e^{i\phi_D} - 1)(X - D')| \\ &= S|X - A'||X - D'|, \end{aligned}$$

the last expression involving the constant $S = |k_A \cdot e^{i\phi_A} - 1| \cdot |k_D \cdot e^{i\phi_D} - 1|$. Thus, the maximal rectangle occurs precisely, when the product $|X - A'||X - D'|$ becomes maximal, for $\{A', D'\}$ fixed and X variable on the circle δ. The proof of the theorem follows then from the following lemma, which should be well known. □

Lemma 6. *For two fixed points $\{A, B\}$ on the circle δ and a variable point X on it, the product $|XA||XB|$ takes an extremal value, precisely when $|XA| = |XB|$. This occurs at the ends $\{O, O'\}$ of the diameter of δ, which is orthogonal to AB.*

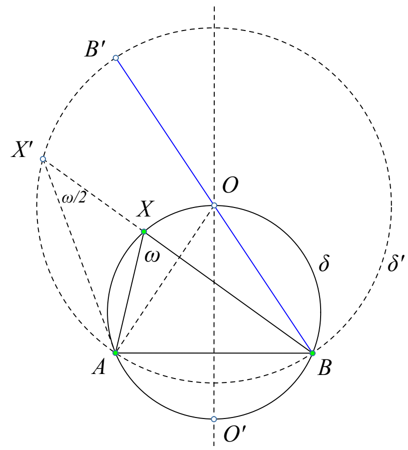

Figure 11. Maximizing the product $|XA||XB|$

Proof. The proof follows immediately from figure 11. Taking $X'X = XA$ on the extension of BX, we see that point X' varies on the circle δ' viewing the segment AB under the angle $\widehat{AX'B} = \widehat{AXB}/2$, whose center O is on the circle δ. Thus, the product $|XA||XB|$ is, up to sign, the power of X w.r. to the circle δ', which becomes maximal, when X coincides with the center O of δ'. If X varies on the other arc defined by the chord AB, then the local maximum occurs analogously at the diametral O' of O. $\square$

Corollary 7. *The similarity ratio of the minimal rectangle to the maximal one is equal, up to sign, to $\tan\left(\frac{\pi}{4} - \frac{\omega}{2}\right)$, where $\omega < \pi/2$ is the angle of the diagonals of the quadrangle of reference $ABCD$.*

Proof. In fact, as was noticed in lemma 3, the similarity ratio of the two extremal rectangles equals, up to sign, the ratio of the sides of the kites $|D'D_2|/|D'D_1| = \tan(\chi/2)$, where $\chi = \widehat{A'D_1D'} = \widehat{A'D_0D'}/2$ (See Figure 12). But the last angle can be readily seen to be equal to $\widehat{A'KD'} = \pi - 2(\phi + \psi) = \pi - 2\omega$, where $\{\phi = \widehat{ADB}, \psi = \widehat{DAC}\}$. $\square$

Combining this with theorem 4, we see that

Corollary 8. *The two extremal rectangles of a non-orthodiagonal quadrangle q are congruent, if and only if the middles of the sides of q are collinear, equivalently its signed area is zero, equivalently its characteristic kites are squares.*

Corollary 9. *The ratio of sides of an extremal rectangle is equal to the ratio of the diagonals of the quadrangle of reference $q = ABCD$.*

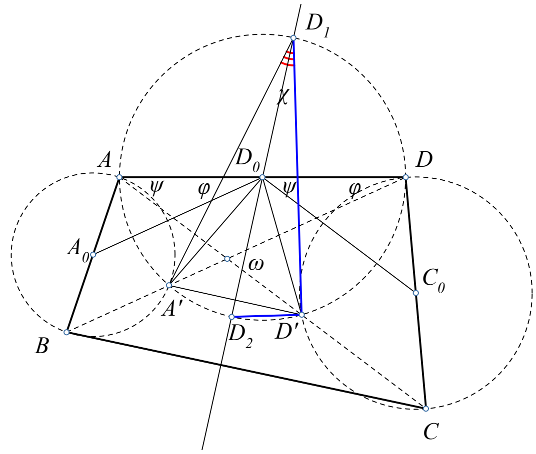

Figure 12. The similarity ratio of the extremal rectangles

Proof. In the proof of the previous theorem we saw that the ratio of the sides of an extremal rectangle can be expressed using complex numbers (see figure-8):

$$\frac{|XY|}{|XZ|} = \frac{|k_A \cdot e^{i\phi_A} - 1|}{|k_D \cdot e^{i\phi_D} - 1|},$$

resulting from the formulas there, in the case of an extremum, for which $|XA'| = |XD'|$. Multiplying this with $|D_0A'| = |D_0D'|$, we see that this ratio expresses the length ratio

$$\begin{aligned} \frac{|k_A \cdot e^{i\phi_A} - 1||D_0A'|}{|k_D \cdot e^{i\phi_D} - 1||D_0D'|} &= \frac{|k_A \cdot e^{i\phi_A}(D_0 - A') - (D_0 - A')|}{|k_D \cdot e^{i\phi_D}(D_0 - D') - (D_0 - D')|} \\ &= \frac{|A_0 - A' - (D_0 - A')|}{|C_0 - D' - (D_0 - D')|} = \frac{|A_0D_0|}{|C_0D_0|}, \end{aligned}$$

which is precisely the claimed ratio of the diagonals. □

Corollary 10. *The extremal rectangles of a quadrangle $ABCD$ are squares, if and only if, the quadrangle has equal diagonals.*

Figure 13 shows cases of quadrangles $ABCD$, whose extremal circumscribed rectangles are squares.

Corollary 11. *The area of the minimal rectangle is zero precisely when the quadrangle of reference $ABCD$ is orthodiagonal.*

Corollary 12. *The positive areas $\{E_b, E_s\}$, respectively of the maximal and minimal rectangles, satisfy the relations*

$$E_b + E_s = |AC||BD| \qquad \text{and} \qquad E_b - E_s = 2E,$$

where E is the non-negative area of the quadrangle of reference $ABCD$.

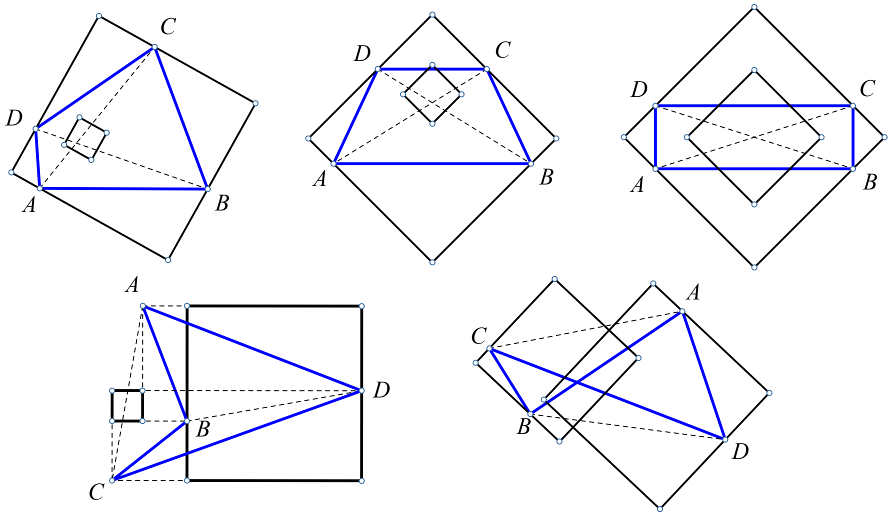

Figure 13. Quadrangles for which the extremal rectangles are squares

Proof. The second relation is already proved in theorem 4, but will be considered here from another view point. In fact, selecting two homologous sides A_1B_1, A_2B_2 of the similar rectangles, the ratio of the areas is

$$\frac{E_s}{E_b} = \frac{|A_2B_2|^2}{|A_1B_1|^2} \quad \Rightarrow \quad \frac{E_s+E_b}{E_b} = \frac{|A_2B_2|^2+|A_1B_1|^2}{|A_1B_1|^2} = \frac{|AC|^2}{|A_1B_1|^2}.$$

Analogously, using the ratio of the two other homologous sides $\{B_1C_1, B_2C_2\}$, we have

$$\frac{E_s}{E_b} = \frac{|B_2C_2|^2}{|B_1C_1|^2} \quad \Rightarrow \quad \frac{E_s+E_b}{E_b} = \frac{|B_2C_2|^2+|B_1C_1|^2}{|B_1C_1|^2} = \frac{|BD|^2}{|B_1C_1|^2}.$$

Since $|A_1B_1||B_1C_1| = E_b$, multiplying the two expressions and simplifying, we get

$$(E_s+E_b)^2 = |AC|^2|BD|^2 \qquad \Leftrightarrow \qquad E_b+E_s = |AC||BD|.$$

On the other side, by corollary 7 and setting $t = \tan(\omega/2)$, where $\omega \le \pi/2$ is the angle of the diagonals of $ABCD$, the ratio of the areas must also be equal to

$$\begin{aligned} \frac{E_s}{E_b} &= \tan\left(\frac{\pi}{4}-\frac{\omega}{2}\right)^2 = \left(\frac{1-t}{1+t}\right)^2 = \frac{1-\sin(\omega)}{1+\sin(\omega)} \\ \Rightarrow & \quad E_b - E_s = (E_b+E_s)\sin(\omega), \end{aligned}$$

which, combined with the first relation, proves the second one. □

Corollary 13. *The positive areas of the extremal rectangles of the quadrangle $ABCD$, whose angle of diagonals $\{AC, BD\}$ is $0 \le \omega \le \pi/2$, are respectively*

$$E_s = \frac{1-\sin(\omega)}{2}|AC||BD|, \qquad E_b = \frac{1+\sin(\omega)}{2}|AC||BD|.$$

Notice that the method used here can be applied also to prove the existence of squares circumscribing a quadrilateral ([1, p.8], [9], see also corollary 29).

5. Two similar orthogonally lying rectangles

Motivated by the configuration of the two extremal rectangles, we study the general case of rectangles $\{\tau_1 = A_1B_1C_1D_1, \tau_2 = A_2B_2C_2D_2\}$, which are similar and have corresponding sides orthogonal or, equivalently, the rotation angle of the similarity has a measure of $\pi/2$. The focus here is on the description of the similarities carrying τ_1 onto τ_2. In general there are two *direct* or *spiral* ([12,

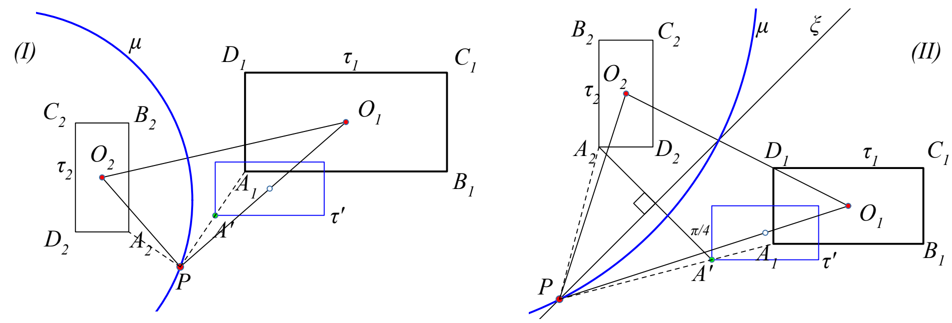

Figure 14. Similarities mapping $A_1B_1C_1D_1$ to $A_2B_2C_2D_2$

p.36,II], [6, p.136]) similarities, and two *antisimilarities* or *dilative reflections* ([12, p.49,II], [6, p.175]) doing this operation. A complete treatment of similarities, including methods to find their centers and other defining them characteristics, can be found in [2, ch.IV], where the direct similarities and the antisimilarities are called respectively *stretch rotations* and *strech reflections*.

Figure 14 gives a short account of the way the similarities are defined in our case. In (I) we have a direct similarity, which per definition is a composition of a homothety and a rotation about the same center P. In (II) we have an antisimilarity, which per definition again is a composition of a homothety and a reflection on a line ξ through the homothety center P. In both cases point P is on the Apollonian circle μ ([1, p.15]), defined as the geometric locus of points X, such that the ratio $XO_2/XO_1 = k$, where k is the homothety ratio and $\{O_1, O_2\}$ are the centers of the similar rectangles, assumed to be different. The figure shows also the intermediate rectangle τ', resulting by applying to τ_1 only the homothety part of the similarity, so that $PA'/PA_1 = k$.

In our configuration, starting from the two similar rectangles $\{\tau_1, \tau_2\}$, the two *direct* similarities $\{f_1, f_2\}$ preserve the orientation and are defined by the correspondence of the vertices suggested by the equations (See Figure 15)

$$f_1(A_1B_1C_1D_1) = D_2A_2B_2C_2 \quad \text{and} \quad f_2(A_1B_1C_1D_1) = B_2C_2D_2A_2.$$

The two *antisimilarities* $\{g_1, g_2\}$, carrying τ_1 onto τ_2 reverse the orientation of the rectangles and correspond their vertices in the order suggested by the equations

$$g_1(A_1B_1C_1D_1) = A_2D_2C_2B_2 \quad \text{and} \quad g_2(A_1B_1C_1D_1) = C_2B_2A_2D_2.$$

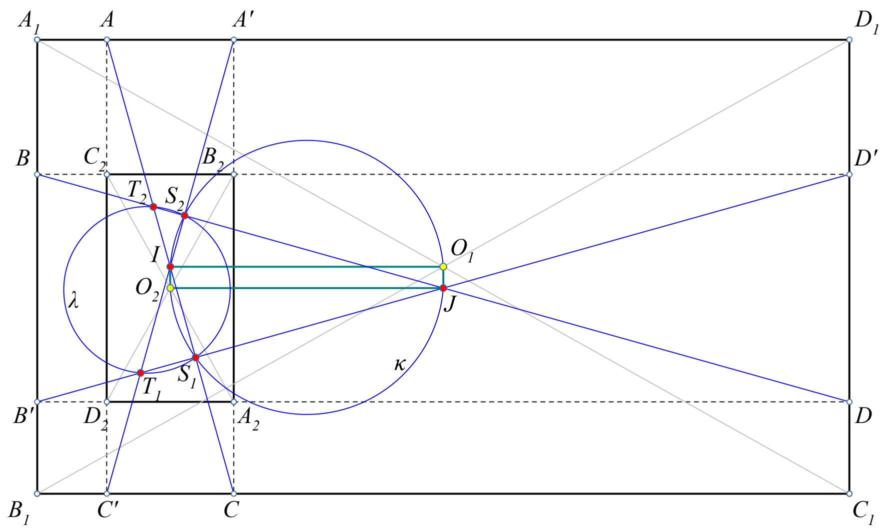

Figure 15. Similarities interchanging $\{\tau_1, \tau_2\}$

The figure shows the locations $\{S_1, S_2\}$ of the two direct- and $\{T_1, T_2\}$ of the two anti-similarities. There are also two rectangles $\{\rho_1 = BB'D'D, \rho_2 = AA'CC'\}$, proved below to be also similar to each other and having the intersections of their diagonals coincident with the similarity centers. They are defined through the intersections of the sides of the given rectangles and will be called *companion rectangles* of $\{\tau_1, \tau_2\}$. There is a sort of symmetry here, since taking the companion rectangles of $\{\rho_1, \rho_2\}$, we come back to $\{\tau_1, \tau_2\}$.

Theorem 14. *With the notation and conventions of this section, the following are valid properties.*

(1) *The centers $\{J, I\}$ of the companion rectangles and the centers $\{O_1, O_2\}$ of $\{\tau_1, \tau_2\}$ are vertices of a rectangle.*
(2) *The two companion rectangles $\{\rho_1, \rho_2\}$ are similar.*
(3) *The similarity centers $\{S_1, S_2\}$ lie on the circumcircle κ of the rectangle IO_1JO_2 symmetrically w.r. to its diameter O_1O_2.*
(4) *The similarity centers $\{S_1, S_2\}$ lie also on respective diagonals of ρ_1, being thus the second intersections of κ with respective diagonals of $\{\rho_1, \rho_2\}$.*

Proof. *Nr-1* is obvious. *Nr-2* follows from the assumed similarity of $\{\tau_1, \tau_2\}$. In fact,

$$\frac{BB'}{B'D} = \frac{CC'}{CA'} \Leftrightarrow \frac{BB'}{CC'} = \frac{B'D}{CA'}.$$

Nr-3 follows from *nr-1* and the fact, that the direct similarities must map the center O_1 to O_2, so that $\widehat{O_1 S_i O_2}$ is a right angle. This shows them to lie on κ. The fact that $S_iO_2/S_iO_1 = k$ is the similarity ratio, makes them symmetric w.r. to the diameter O_1O_2 of κ.

Nr-4 follows from *nr-3*, since this implies that $\widehat{S_1JS_2}$ is bisected by JO_2, which is parallel to BD'. On the other side we have also that $BB'/BD' = k$. This implies that $\{S_1, J, D'\}$ are collinear. Analogously is seen that $\{S_2, J, D\}$ are collinear. By the similarity of $\{\rho_1, \rho_2\}$ follows then that the similarity centers are respectively the intersections $S_1 = (B'D, AC), S_2 = (BD, A'C')$. □

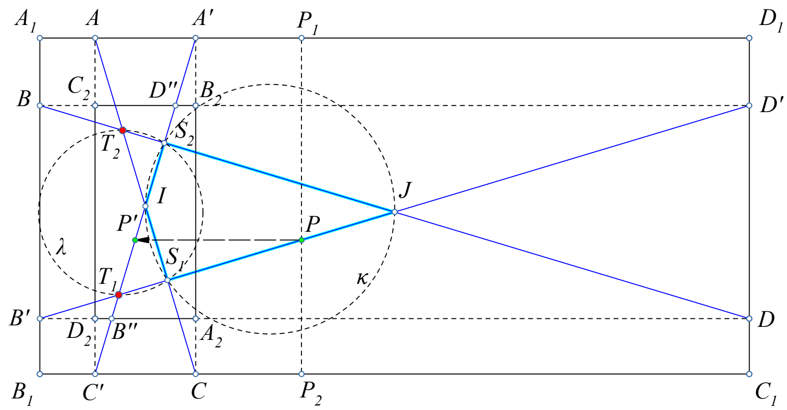

Figure 16. The centers $\{T_1, T_2\}$ of the antisimilarities

Theorem 15. *Continuing with the notation and conventions adopted so far, the following are valid properties.*

(1) *The centers $\{T_1, T_2\}$ of the antisimilarities $\{g_1, g_2\}$ are the intersection points of the opposite sides of the cyclic quadrangle IS_1JS_2.*
(2) *Points $\{T_1, T_2\}$ are diametral points of a circle λ, which is orthogonal to the circumcircle κ of IS_1JS_2 and passes through $\{S_1, S_2\}$.*

Proof. For *nr-1* we work with the similarity g_1, showing that the intersection point $T_1 = (A'C', B'D')$ is its similarity center. The proof for the similarity center T_2 of g_2 is completely analogous. We start by showing that $B'D'$ maps under g_1 onto line $A'C'$. In fact, since g_1, by definition, maps line D_1C_1 onto line B_2C_2, point $D' \in D_1C_1$ will map to some point $D'' \in B_2C_2$. Analogously, since line A_1B_1 maps onto line A_2D_2, point $B' \in A_1B_1$ will map to some point $B'' \in A_2B_2$. By the preservation of ratios by similarities follows

$$\frac{D'D_1}{D'C_1} = \frac{B_2A'}{B_2C} = \frac{D''B_2}{D''C_2} = \frac{B_2A'}{B_2C} \quad \Rightarrow \quad \frac{B_2D''}{B_2A'} = \frac{D''C_2}{B_2C} = \frac{D''C_2}{C_2C'}.$$

This implies that D'' is on $A'C'$ and analogously B'' is also on this line, hence g_1 maps line $D'B'$ onto line $A'C'$, as claimed. Now, projecting P to points $\{P_1, P_2\}$ on the parallels $\{A_1D_1, B_1C_1\}$ and working analogously with the ratios PP_1/PP_2 of the varying point $P \in D'B'$ and its image $P' = g_1(P)$ (See Figure 16), we see that line PP' is always parallel to A_1D_1, hence T_1 is the fixed point of g_1.

Nr-2 is a trivial consequence of the previous properties, since $\{\widehat{T_2S_2I}, \widehat{T_1S_1I}\}$ are right angles. It is also a general property for cyclic quadrangles. This is handled in detail in [11], where the circle λ on diameter KL is called the *orthocycle* of the cyclic quadrangle IS_1JS_2, here coinciding also with an Apollonian circle of the segment O_1O_2. □

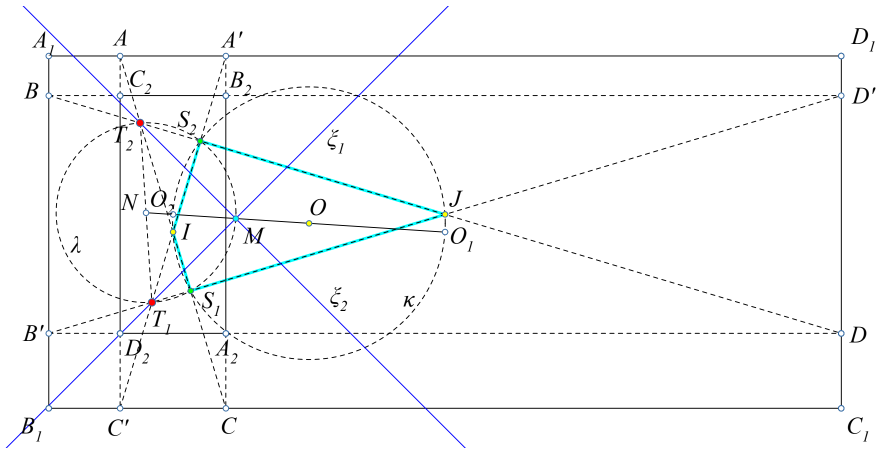

Figure 17. The axes $\{T_1M, T_2M\}$ of the antisimilarities $\{g_1, g_2\}$

Regarding the *axes* $\{\xi_1, \xi_2\}$ of the reflections involved in the definition of the antisimilarities, the following theorem describes their location. The proof uses the fact that such an axis passes through the center of the antisimilarity and is bisecting the angle of a line through its center and its image-line under the antisimilarity (See Figure 17).

Theorem 16. *Continuing with the notation and conventions adopted so far, the following are valid properties.*

1. *The axes of the antisimilarities $\{g_1, g_2\}$ are respectively bisectors of the angles $\{\widehat{JT_1I}, \widehat{IT_2J}\}$.*
2. *These lines intersect orthogonally at a point M on line O_1O_2, which passes also through the center N of the circle λ.*
3. *The four points $\{T_1, T_2, I, J\}$ define an orthocentric quadruple, i.e. each triple of them defines a triangle whose orthocenter is the fourth point.*

Proof. Nr-1 derives directly from the definition of the antisimilarity and the fact proved in the previous theorem, that line $B'D'$ maps under g_1 onto line $A'C'$. This shows that the bisector $\xi_1 = T_1M$ of the angle $\widehat{S_1T_1S_2}$ is the axis of g_1. Analogously is seen that the bisector $\xi_2 = T_2M$ of the angle $\widehat{S_1T_2S_2}$ is the axis of g_2.

Nr-2, the part of orthogonality $\xi_1 \perp \xi_2$, results by a simple angle chasing argument and is left as an exercise ([3, p.21]). Because $\xi_1 = T_1M$ is a bisector of the angle $\widehat{S_1T_1S_2}$, point M is the middle of the arc S_1S_2 of the circle λ. This implies the other claims of this *nr*.

Nr-3 derives from the fact that $\{T_1S_2, T_2S_1\}$ are two altitudes of triangle T_2T_1J, intersecting at I. □

We call the ordered orthocentric quadruple (T_1, T_2, I, J) the *associated quadruple* of $\{\tau_1, \tau_2\}$. By the previous theorems, the four similarities carrying τ_1 onto τ_2 can be completely determined by the data of this quadruple. Also, given such a quadruple, and setting I as the orthocenter of the triangle T_1T_2J, we can define a double infinity of configurations like the one of figure 17, by selecting arbitrarily the position of one vertex of τ_1, like the point C_1 say. In fact, using the quadruple, we can easily determine the circle κ and its diameter O_1O_2. Then, reflecting the arbitrary point C_1 on the lines $\{O_1I, O_1J\}$ we define respectively the points $\{D_1, B_1\}$ and from these the rectangle $\tau_1 = A_1B_1C_1D_1$. Then, we define the direct similarity f_1 with center at S_1, angle $\pi/2$ and ratio S_1O_2/S_1O_1 and through it the rectangle $\tau_2 = f_1(\tau_1) = A_2B_2C_2D_2$. The two rectangles $\{\tau_1, \tau_2\}$ define by the procedures of this section an associated quadruple coinciding with the given one. The four similarities mapping τ_1 to τ_2 are in all cases the same. Also their companion rectangles are all similar to each other and are characterized by the angle of their diagonals, which is $\widehat{T_1JT_2}$. We summarize these facts in the form of the next corollary.

Corollary 17. *Every ordered orthocentric quadruple (T_1, T_2, I, J) with point I selected as orthocenter of the triangle T_1T_2J, defines a diameter O_1O_2 on the circle with diameter IJ and a double infinity of similar rectangles $\{\tau_1, \tau_2\}$, centerred correspondingly at $\{O_1, O_2\}$ and with sides parallel to $\{O_1I, O_1J\}$, such that the associated quadruple is the given one and all their companion rectangles $\{\rho_1, \rho_2\}$ are similar, having the same angle of diagonals $\widehat{T_1JT_2}$.*

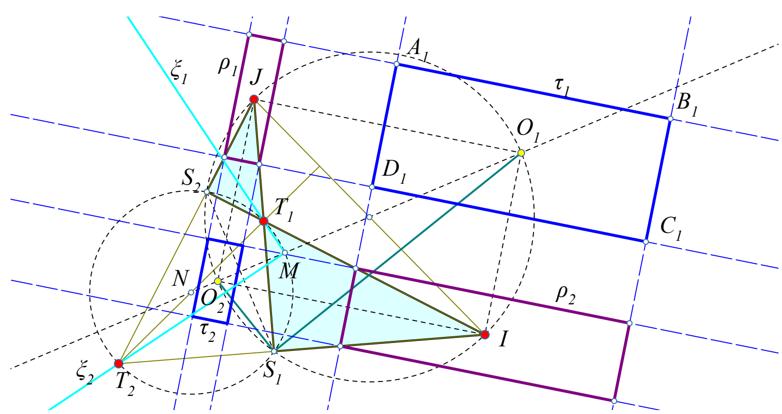

Figure 18. Similar rectangles $\{\tau_1, \tau_2\}$ and their companions $\{\rho_1, \rho_2\}$

Since an unordered quadruple defines six ordered pairs of the type (T_1, T_2, I, J) we obtain six possibilities to construct such double infinities of similar rectangles and their companions. Figure 18 shows a case in which the quadrangle IS_1JS_2 is self intersecting. Notice that the four similarities and their inverses, interchanging $\{\tau_1, \tau_2\}$, do not map the companions $\{\rho_1, \rho_2\}$ to each other. Latter rectangles are interchanged by four other similarities and their inverses, creating an analogous

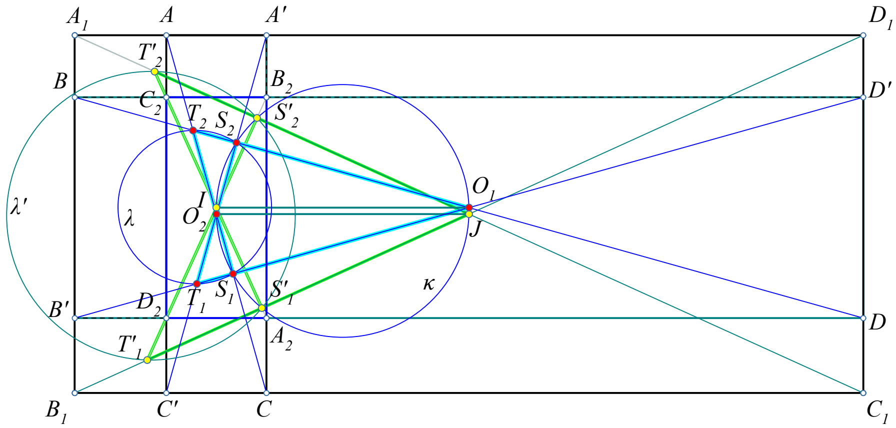

Figure 19. Similarities interchanging the companions

orthocentric quadruple (T_1', T_2', O_1, O_2) (See Figure 19). The figure reflects the aforementioned symmetry, by which, taking the companions of $\{\rho_1, \rho_2\}$, we come back to $\{\tau_1, \tau_2\}$. The similarity centers $\{T_1', T_2', S_1', S_2'\}$, in this case, are on an Apollonian circle λ' of the segment IJ, which, like λ, is orthogonal to κ carrying $\{S_1, S_2, S_1', S_2'\}$. It is easily seen that also $\{T_1, T_2, T_1', T_2'\}$ are on a circle κ'.

6. The case of the quadrangle and its twin

Leaving aside, for a while, the special case of the parallelogram and considering a generic non-orthodiagonal quadrangle $q = ABCD$, we formulate first the consequences of the results of the previous section for the pair of its extremal rectangles $\{\tau_1, \tau_2\}$.

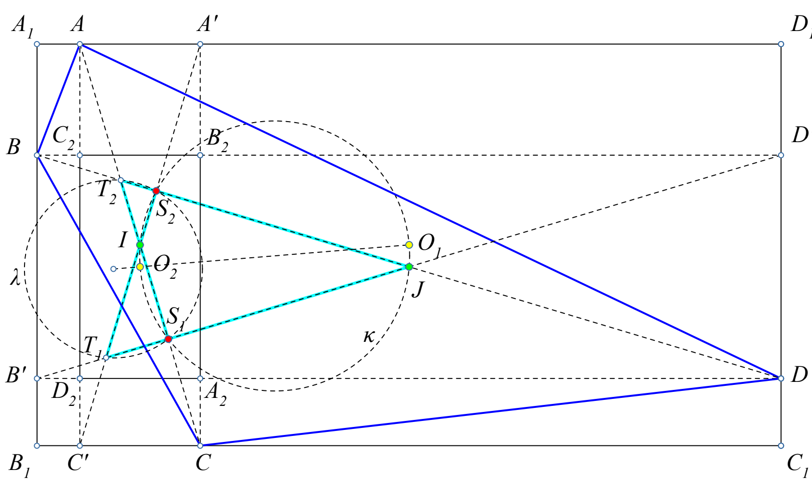

Figure 20. Similarity centers of extremal rectangles of $q = ABCD$

Theorem 18. *With the notation and conventions introduced so far, the following are valid properties for the generic quadrangle $q = ABCD$.*

(1) *The centers $\{O_1, O_2\}$ of the rectangles, respectively, $\{\tau_1, \tau_2\}$ and the middles $\{I, J\}$ of the diagonals of q define a rectangle IO_1JO_2, with sides parallel to the sides of the extremal rectangles.*
(2) *The direct similarity centers $\{S_1, S_2\}$ lie on the circumcircle κ of the previous rectangle with diameter IJ and define a kite $S_1O_1S_2O_2$, which is similar to the kites carrying the vertices of $\{\tau_1, \tau_2\}$.*
(3) *The similarity centers $\{S_1, S_2\}$ lie also on respective diagonals of q, being thus the second intersections of κ with the diagonals of q.*
(4) *The centers $\{T_1, T_2\}$ of the antisimilarities are the intersection points of the opposite sides of the cyclic quadrangle IS_1JS_2.*
(5) *Points $\{T_1, T_2\}$ are diametral points of a circle λ, which is orthogonal to the circumcircle κ of IS_1JS_2, contains also the centers $\{S_1, S_2\}$ and coincides with the Apollonian circle of the segment O_1O_2 for the ratio $k = \tan(\pi/4 - \omega/2)$, where $0 < \omega < \pi/2$ is the angle of the diagonals of q.*

Corollary 19. *The centers of the two direct similarities, relating the extremal rectangles, are the projections of the middles of the diagonals of the quadrangle on the other diagonals.*

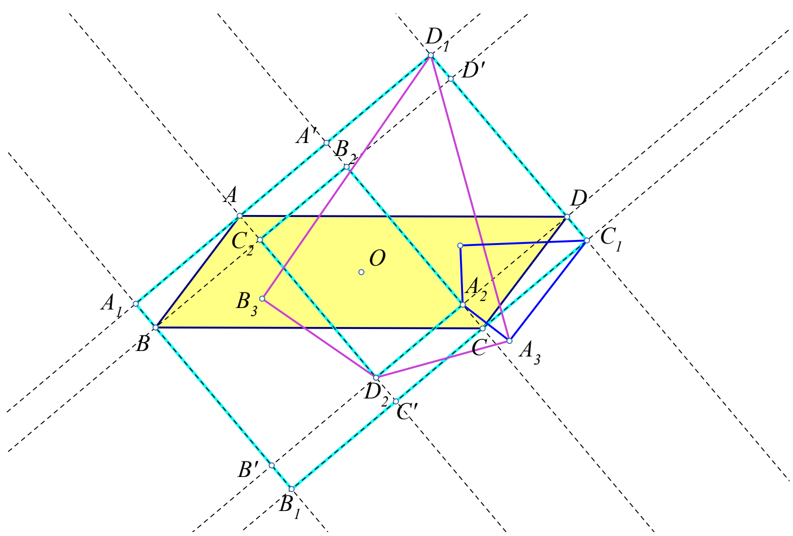

Figure 21. The case of parallelograms

Figure 21 shows the left aside case of a parallelogram $q = ABCD$, displaying its two extremal rectangles and two of the characteristic kites, defining them. All the similarity centers of the direct- as well the anti-similarities here coincide with the center of the parallelogram. The proof of the next corollary is left as an exercise.

Corollary 20. *In the case of a parallelogram $q = ABCD$, the extremal rectangles have their centers, as well as, the centers of the similarities, coinciding with the center O of q. The ratio of the similarities is, as in the generic case, expressible through the sides of the kites, $k = |D_2A_3|/|D_1A_3| = \tan(\pi/4 - \omega/2)$, where $0 < \omega < \pi/2$ is the angle of the diagonals of q.*

Given the non-orthodiagonal quadrangle $q = ABCD$, we can define a *twin* quadrangle $q' = A'B'C'D'$, which, like q, is also inscribed simultaneously in the associated extremal rectangles $\{\tau_2, \tau_1\}$ of q and has the same companion rectangles. This is seen in figure 22, which among other properties shows that the new

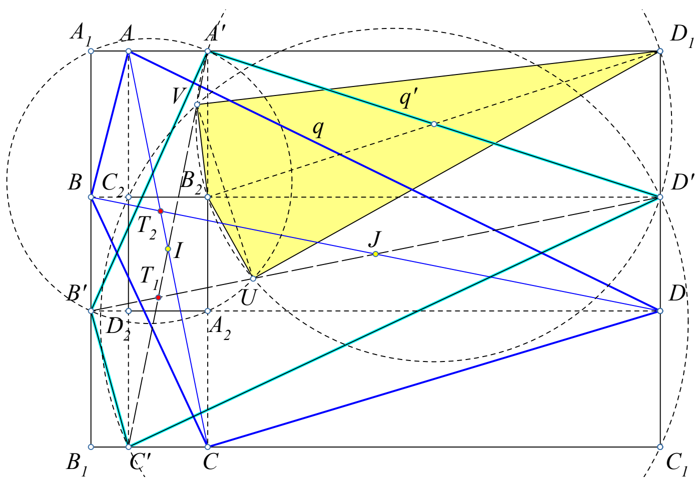

Figure 22. The "twin" quadrangle $A'B'C'D'$ of $ABCD$

quadrangle has diagonals of the same length and same angle between them with the original one, hence also the same area. In addition, the two rectangles are also extremal w.r. to the new one and the orthocentric quadruple (T_1, T_2, I, J) plays the same role for q' as it does for q. The characteristic property of q' is that its diagonals coincide with the diagonals of the companion rectangles $\{AA'CC', BB'DD'\}$, which are different from the diagonals of q. Figure 22 displays also a characteristic kite for q' carrying vertices of its own extremal rectangles and seen to be identical with D_1 and B_2. In fact, the two circles on diameters, respectively, $\{A'D', A'B'\}$ intersect at a point U of $B'D'$, which defines the altitude $A'U$ of triangle $A'B'D$. Analogously the circles on diameters $\{A'D', D'C'\}$ intersect at point V defining the altitude DV of triangle $A'D'C'$. From the similarity of the rectangles $\{AA'CC', BB'DD'\}$ follows that

$$\widehat{B_2D_1V} = \widehat{B_2A'V} = \widehat{B_2D'U} = \widehat{B_2D_1U},$$

which proves that VD_1UB_2 is a kite of q', like those carrying the vertices of the extremal rectangles, considered in section 2. Next theorem summarizes these observations.

Theorem 21. *For every non-orthodiagonal quadrangle q, the associated twin quadrangle q' shares with q the same extremal rectangles and the same associated orthocentric quadruple.*

Notice that the kites of q', like the VD_1UB_2 of figure 22, though not identical to those of q, they are nevertheless similar to them, since their similarity type is completely determined by the angle of the diagonals of q', which is the same with that of q. Notice also that the "twin" relation is reflective, so that the twin of q' is the original quadrangle q. Also, using corollary 19, we deduce easily the follow-

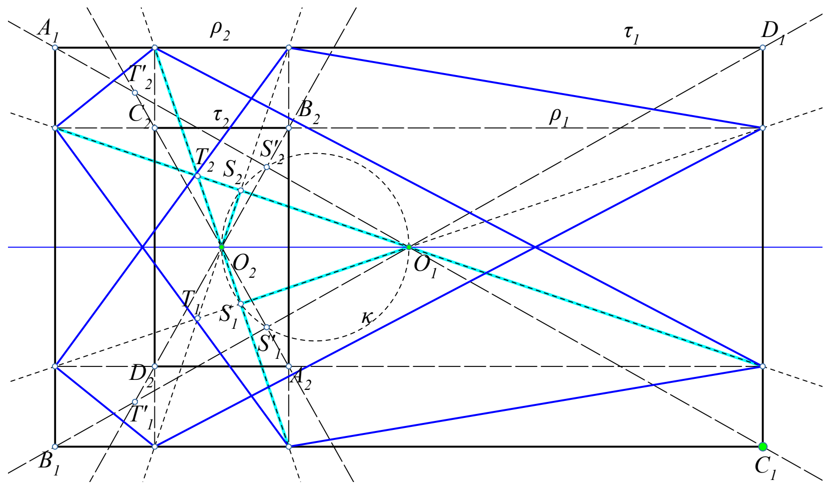

Figure 23. Quadrangle with congruent twin

ing characterizations of the particular class of quadrangles, which have congruent twins (See Figure 23).

Corollary 22. *The twin quadrangle q' of a non-orthodiagonal, non-parallelogrammic quadrangle q is congruent to q precisely when the projections of the middles of the diagonals of q on the other diagonals are symmetric w.r. to the Newton line, joining these middles, equivalently, the sides of the extremal rectangles are respectively parallel and orthogonal to the Newton line, equivalently, the centers of these rectangles coincide with the middles of the diagonals of q, equivalently the twin q' is the reflection of q on the Newton line.*

An easy testing, which I omit, of the various possibilities to define a quadrangle simultaneous inscribed in two given similar and orthogonally lying rectangles, shows the following corollary.

Corollary 23. *Given two similar and orthogonally lying rectangles $\{\tau_1, \tau_2\}$, there is precisely one pair of twin quadrangles having them for extremal.*

7. The associated orthodiagonals

An additional feature of the existence of the two extremal rectangles is the existence of a couple of orthodiagonal quadrangles inscribed simultaneously in these two rectangles. Next corollary, whose easy proof is left as an exercise, summarizes their characteristics.

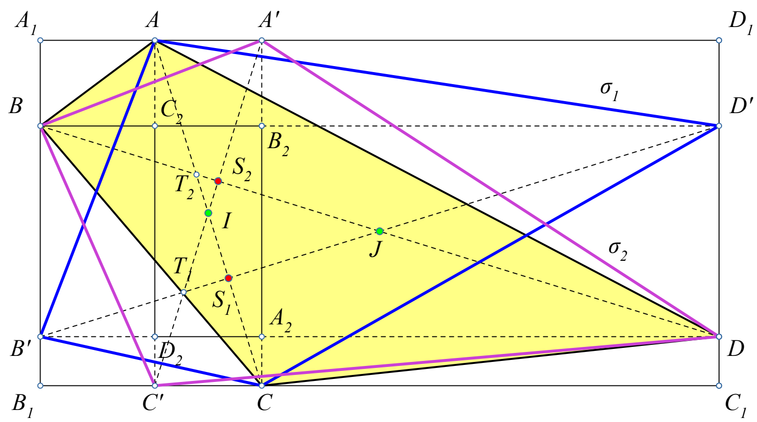

Figure 24. The two orthodiagonals $\{\sigma_1, \sigma_2\}$ defined by a non-orthodiagonal $ABCD$

Corollary 24. *Every non-orthodiagonal quadrangle $q = ABCD$ defines two orthodiagonal quadrangles $\{\sigma_1 = AB'CD', \sigma_2 = A'BC'D\}$, which are inscribed in the two extremal quadrangles $\{\tau_1, \tau_2\}$ of q. The quadrangle $\sigma_1(\sigma_2)$ shares with q the diagonal $AC(BD)$ and its other diagonal $|B'D'| = |BD|(|A'C'| = |AC|)$. The intersection points of the diagonals of $\{\sigma_1, \sigma_2\}$ coincide with the similarity centers $\{S_1, S_2\}$, and the areas of these orthodiagonals are equal to the difference $E_b - E$. In the case q is a parallelogram, $\{\rho_1, \rho_2\}$ are rhombi lying symmetric w.r. to its center O.*

An easily proved consequece of these observations is the following corollary.

Corollary 25. *The sum of squares of the sides of the generic non-orthodiagonal quadrangle q is equal to the corresponding sum of squares of its twin quadrangle q'.*

8. The case of orthodiagonals

In the case $ABCD$ is an orthodiagonal quadrangle the following theorem lists several related facts, which more or less are well known and their proof is left as an exercise on the ground of figure 25. In this, points $\{I, J\}$ are the middles of the diagonals intersecting at point K, points $\{M, L\}$ are the other than K intersections respectively of the circles $\{\beta \cap \delta, \alpha \cap \gamma\}$ and $\{H, N\}$ are respectively the centers of the maximal circumscribing $A_1B_1C_1D_1$ and the variable circumscribing rectangle $A^*B^*C^*D^*$.

Theorem 26. *Every orthodiagonal quadrangle $ABCD$ has the following properties.*

(1) *All circumscribing rectangles are similar to each other and the maximal one $A_1B_1C_1D_1$ has its sides parallel to the diagonals of $ABCD$.*

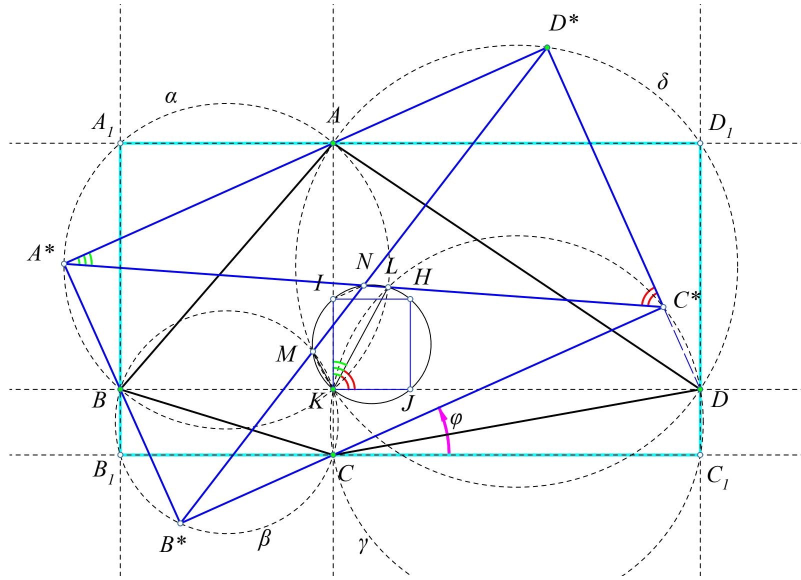

Figure 25. Circumsribing rectangles of an orthodiagonal quadrangle $ABCD$

(2) *The similarity center of two such rectangles is at point K and the similarity ratio of the variable rectangle $A^*B^*C^*D^*$ to the maximal one $A_1B_1C_1D_1$ is $\cos(\phi)$, where ϕ is the angle between two homologous sides of these two rectangles.*

(3) *$KJHI$ is a rectangle and points $\{L, M, N\}$ lie on its circumcircle.*

Corollary 27. *A quadrangle is orthodiagonal, if and only if all its circumscribed rectangles are similar to the rectangle of its diagonals, equivalently, if all its circumscribed rectangles are similar to each other.*

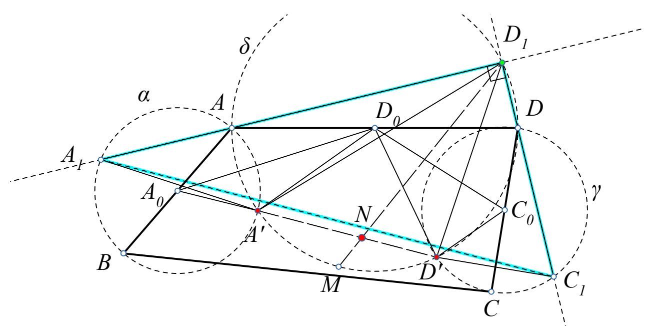

Figure 26. The ratio of sides of circumscribing rectangles

Proof. Here the necessity part follows from the previous theorem. The sufficiency is also easily deduced from figure 26 and using the arguments of section 3, by which the ratio of the sides of the circumscribed rectangle is

$$\frac{D_1A_1}{D_1C_1} = S \cdot \frac{D_1A'}{D_1D'} = S \cdot \frac{NA'}{ND'}.$$

Here S is a constant and N is the trace on $A'D'$ of the bisector of the angle $\widehat{A'D_1D'}$. In the case the quadrangle of reference is non-orthodiagonal, the last ratio cannot be constant for D_1 varying on δ, hence the proof of the sufficiency of the corollary. $\square$

A refinement of the previous argument, considering three positions of N for which the ratio NA'/ND' has the constant value k or $1/k$, proves also next corollarylaries.

Corollary 28. *A quadrangle is orthodiagonal, if and only if it has three similar circumscribed rectangles. In this case all its circumscribed rectangles are similar.*

Corollary 29. *A quadrangle has equal and orthogonal diagonals, if and only if it has three circumscribed squares. In this case all its circumscribed rectangles are squares.*

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