Forum Geometricorum
Volume 18 (2018) 185–194.

FORUM GEOM
ISSN 1534-1178

The Blundon Theorem in an Acute Triangle and Some Consequences

Mihály Bencze and Marius Drăgan

Abstract. The purpose of this article is to give an analoque of Blundon theorem in an acute triangle and using this result to obtain the best inequality of the type

$$\sum \sqrt{\frac{b+c-a}{a}} \geq f(R,r)$$

where f is a homogenous function.

Let $C(O,r)$ and $C(I,r)$ be two circles such that $I \in int\, C(O,r)$ and $OI = \sqrt{R^2 - 2Rr}$.

For any triangle ABC with sides $BC = a$, $CA = b$, $AB = c$, and semiperimeter $s = \frac{a+b+c}{2}$, we denote by $C(O,R)$ the circumcircle and $C(I,r)$ the incircle.

Theorem of the present paper is an analogue in an acute triangle of Theorem 2 of Blundon [3].

Also Theorem represents the best improvement of the type

$$\sum \sqrt{\frac{b+c-a}{a}} \geq f(R,r),$$

where $f(R,r)$ is a homogeneous function of the inequality $\sum \sqrt{\frac{b+c-a}{a}} \geq 3$. See [1, p. 159-165], which is known as the Rădulescu - Maftei Theorem and which in [1] has 2 solutions, one elementary and other based on the Lagrange multiplier Theorem.

Main Results

Lemma 1. *In any triangle ABC are true the following equalities*

1). $a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right)$

2). $ab + bc + ca = s^2 + r^2 + 4Rr$

3). $a^2b^2 + b^2c^2 + c^2a^2 = \left(s^2 + r^2 + 4Rr\right)^2 - 16Rrs^2$

Lemma 2. *In any triangle ABC is true the following equality:*

Publication Date: June 1, 2018. Communicating Editor: Paul Yiu.

$$\prod \cos A = \frac{s^2 - r^2 - 4Rr - 4R^2}{4R^2}$$

Proof. In the following we will denote $x = a^2 + b^2 + c^2$. From the cosine theorem it follows that:

$$\prod \cos A = \frac{\prod \left(b^2 + c^2 - a^2\right)}{8(\prod a)^2} = \frac{\prod \left(x - 2a^2\right)}{8(\prod a)^2} =$$

$$= \frac{x^2 - 2\sum a^2 x + 4\sum a^2 b^2 x - 8(\prod a)^2}{8(\prod a)^2} = \frac{s^2 - r^2 - 4Rr - 4R^2}{4R^2}$$

□

Theorem 3. *In any acute triangle is true the following inequality:*

$$s > 2R + r$$

Proof. As in any acute triangle is true the inequality: $\prod \cos A > 0$ according with Lemma 2 it follows the inequality from the statement. □

Theorem 4 (Blundon)**.** *In any triangle ABC is true the following inequality: $s_1 \leq s \leq s_2$ where*

$$s_1 = \sqrt{2R^2 + 10Rr - r^2 - 2\sqrt{R\left(R - 2r\right)^3}}\,,\ s_2 = \sqrt{2R^2 + 10Rr - r^2 + 2\sqrt{R\left(R - 2r\right)^3}}$$

represent the semiperimeter of two issoscels triangle $A_1B_1C_1$ and $A_2B_2C_2$ with the sides

$$a_1 = 2\sqrt{R^2 - (r - t)^2},\ b_1 = c_1 = \sqrt{2R\left(R + r - t\right)}$$

$$a_2 = 2\sqrt{R^2 - (r + t)^2},\ b_2 = c_2 = \sqrt{2R\left(R + r + t\right)}$$

where $t = OI = \sqrt{R^2 - 2Rr}$.

Lemma 5. *Let $A_3B_3C_3$ be a triangle with $C\left(O, R\right)$ the circumscrible and $C\left(I, r\right)$ the incircle and with the semiperimeter $s_3 = 2R + r$. Then the sides of triangle $A_3B_3C_3$ is unique determinated by the equalities:*

$$a_3 = 2R$$

$$b_3 = R + r + \sqrt{R^2 - 2Rr - r^2}$$

$$c_3 = R + r - \sqrt{R^2 - 2Rr - r^2}$$

where A_3 is a right angle.

Proof. We have the following equalities:

$$a+b+c=2s$$

$$ab+bc+ca=s^2+r^2+4Rr$$

$$abc=4Rrs$$

or

$$a+b+c=4R+2r$$

$$ab+bc+ca=4R^2+8Rr+2r^2 \quad (1)$$

$$abc=4Rr\left(2R+r\right)$$

From (1) it follows that a,b,c are the solutions of the equation:

$$u^3-\left(4R+2r\right)u^2+\left(4R+8Rr+2r^2\right)u-4Rr\left(2R+r\right)=0 \quad (2)$$

The equation (2) may be written as:

$$\left(u-2R\right)\left[u^2-\left(2R+2r\right)u+4Rr+2r^2\right]=0$$

which has the solutions from the statement. □

Theorem 6. *In any acute triangle with $C\left(O,R\right)$ the circumscribed and $C\left(I,r\right)$ the inscribed are true the following inequalities:*

$$s_1\leq s\leq s_2 \text{ if } 2\leq \frac{R}{r}<\sqrt{2}+1$$

and

$$s_3\leq s\leq s_2 \text{ if } \frac{R}{r}\geq \sqrt{2}+1$$

where s_1,s_2 are the semiperimeter of two issosceles triangle $A_1B_1C_1,A_2B_2C_2$ with the sides from Theorem 2 and s_3 is the semiperimeter of the right triangle $A_3B_3C_3$ from Lemma 3.

Proof. We denote $\frac{R}{r}=x$. We consider two cases:

Case 1. $2\leq x<\sqrt{2}+1$

We will prove that $s_1>s_3$ or in an equivalent form:

$$2x^2+10x-1-2\sqrt{x\left(x-2\right)^3}-\left(2x+1\right)^2=2\left[-\sqrt{x\left(x-2\right)^3}-\left(x^2-3x+1\right)\right]>0$$

or

$$-\left(x^2-3x+1\right)>\sqrt{x\left(x-2\right)^3} \quad (3)$$

But $x^2-3x+1<0$ as $x<\sqrt{2}+1<\frac{3+\sqrt{5}}{2}$. After squaring in (3) we obtain:

$$\left(x^2-3x+1\right)^2>x\left(x-2\right)^3 \text{ or } -x^2+2x+1>0 \text{ or}$$

$$\left(\sqrt{2}-1-x\right)\left(x-\left(\sqrt{2}+1\right)\right)>0$$

inequality which is true. It results that $s_3<s_1\le s_2$.

But as $s_1\le s\le s_2$ and $s\ge s_3$ it follows that $s_1\le s\le s_2$.

Case 2a. $\sqrt{2}+1\le x<\frac{3+\sqrt{5}}{2}$ or $x^2-3x+1<0$.

We will prove that $s_1\le s_3$ or in an equivalent form:

$$2x^2+10x-1-2\sqrt{x\left(x-2\right)^3}<\left(2x+1\right)^2 \text{ or } -\left(x^2-3x+1\right)\le\sqrt{x(x-2)^3} \quad (4)$$

After squaring and performing some calculation the inequality (4) may be written as

$$\left(x-\left(\sqrt{2}-1\right)\right)\left(x-\left(\sqrt{2}+1\right)\right)\ge 0$$

inequality which is true.

We will prove that $s_3<s_2$ or in an equivalent form:

$$\left(2x+1\right)^2<2x^2+10x-1-2\sqrt{x\left(x-2\right)^3} \quad \text{or} \quad x^2-3x+1<\sqrt{x\left(x-1\right)^3} \quad (5)$$

The inequality (5) is true as $x^2-3x+1<0$. It results that $s_1\le s_3<s_2$. But as $s_1\le s\le s_2$ and $s\ge s_3$ it follows that $s_3\le s\le s_2$.

Case 2.b. $x\ge\frac{3+\sqrt{5}}{2}$ or $x^2-3x+1\ge 0$.

We will prove that

$$s_1<s_3 \text{ or } -\left(x^2-3x+1\right)<\sqrt{x\left(x-2\right)^3}$$

inequality which is true.

We will prove that

$$s_3<s_2 \text{ or } x^2-3x+1<\sqrt{x\left(x-2\right)^3}$$

or in an equivalent form

$$\left[x-\left(\sqrt{2}-1\right)\right]\left[x-\left(\sqrt{2}+1\right)\right]>0$$

It results that $s_1<s_3<s_2$. But as $s_1\le s\le s_2$ and $s\ge s_3$ it follows that $s_3\le s\le s_2$.

It results in the cases 2a and 2b that $s_3\le s\le s_2$ which is equivalent with the inequality from the statement. $\square$

Lemma 7. *In any triangle ABC is true the equalities:*

1). $\sum \frac{s-a}{a} = \frac{s^2+r^2-8Rr}{4Rr}$

2). $\sum \frac{(s-a)(s-b)}{ab} = \frac{2R-r}{2R}$

Proof.

$$\sum \frac{s-a}{a} = \frac{s\sum bc - 3abc}{abc} = \frac{s\left(s^2+r^2+4Rr\right) - 12Rr}{abc} = \frac{s^2+r^2-8Rr}{4Rr} =$$

$$= \sum \frac{(s-a)(s-b)}{ab} = \frac{s^2\left(\sum a\right) - 2s\left(s^2+r^2+4Rr\right) + 12Rrs}{abc} = \frac{2R-r}{2R}$$

□

Theorem 8 (A refinement of Rădulescu - Maftei Theorem)**.** *In any triangle ABC is true the following inequality:*

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{2R-2\sqrt{R^2-2Rr-r^2}}{R+r+\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{2R+2\sqrt{R^2-2Rr-r^2}}{R+r-\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{r}{R}}$$

if $\frac{R}{r} \geq \sqrt{2}+1$ *or*

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{R-r-d}{r}} + 2\sqrt{\frac{R+d}{R}}$$

if $2 \leq \frac{R}{r} < \sqrt{2}+1$.

Proof. We denote $t = \sum \sqrt{\frac{s-a}{a}}$. By squaring we obtain

$$t^2 = \sum \frac{s-a}{a} + 2\sqrt{\frac{\sum (s-a)(s-b)}{ab} + 2\sqrt{\frac{(s-a)(s-b)(s-c)}{abc}}}$$

From Lemma 4, 1) and 2) it follows that:

$$\left(t^2 - \frac{s^2+r^2-8Rr}{4Rr}\right)^2 = 4\left(\frac{2R-r}{2R} + 2\sqrt{\frac{r}{4R}}t\right)$$

We consider the function $f:(0,+\infty) \to R$

$$f(u) = u^4 - \frac{s^2+r^2-8Rr}{2Rr}u^2 - 8\sqrt{\frac{r}{4R}}u + \left(\frac{s^2+r^2-8Rr}{4Rr}\right)^2 - \frac{4R-2r}{R}$$

We have $f(t) = 0$. We will prove that

$$\left(\frac{s^2+r^2-8Rr}{4Rr}\right)^2 < \frac{4R-2r}{R}$$

or in an equivalent form:

$$s^2 < 8Rr - r^2 + 4\sqrt{Rr^2(4R-2r)}$$

But as $s^2 \leq s_2^2$. It will be sufficient to prove that

$$s_2^2 = 2R^2 + 10Rr - r^2 - 2\sqrt{R\left(R-2r\right)^3} < 8Rr - r^2 + 4\sqrt{Rr^2\left(4R-2r\right)} \quad (6)$$

We denote $x = \frac{R}{r}$. The inequality (6) may be written as:

$$2x^2 + 10x - 1 - 2\sqrt{x\left(x-2\right)^3} < 8x - 1 + 4\sqrt{x\left(4x-2\right)}$$

or

$$x^2 + x < \sqrt{x\left(x-2\right)^3} + 2\sqrt{x\left(4x-2\right)} \quad (7)$$

After squaring the inequality (7) we will obtain:

$$x^4 + 2x^3 + x^2 < x\left(x^3 - 6x^2 + 12x - 8\right) + 16x^2 - 8x + 4x\sqrt{\left(x-2\right)^3\left(4x-2\right)}$$

or

$$8x^3 - 27x^2 + 16x < 4x\sqrt{\left(x-2\right)^3\left(4x-2\right)}$$

or

$$8x^2 - 27x + 16 < 4\sqrt{\left(x^3 - 6x^2 + 12x - 8\right)\left(4x-2\right)} \quad (8)$$

If

$$8x^2 - 27x + 16 \leq 0$$

the inequality (8) is true. For $8x^2 - 27x + 16 > 0$ we will square (8) and we will obtain:

$$64x^4 + 729x^2 + 256 - 432x^3 + 256x^2 - 864x < 64x^4 - 416x^3 + 960x^2 - 896x + 256$$

or

$$16x^3 - 25x^2 - 32x > 0 \text{ or } 16x^2 - 25x - 32 > 0$$

But $8x^2 - 27x + 16 > 0$. It results that $x > \frac{27+\sqrt{217}}{16} > \frac{25+\sqrt{2673}}{32}$ or $16x^2 - 2x - 32 > 0$.

We denote $a_2 = \frac{s^2+r^2-8Rr}{2Rr}$, $a_1 = 8\sqrt{\frac{r}{4R}}$, $a_0 = \frac{4R-2r}{R} - \left(\frac{s^2+r^2-8Rr}{4Rr}\right)^2$. The equation $f\left(u\right) = 0$ may be written as: $u^4 - a_2u^2 - a_1u - a_0 = 0$ with $a_0, a_1, a_2 > 0$ or $1 - \frac{a_2}{u^2} - \frac{a_1}{u^3} - \frac{a_0}{u^4} = 0$. But $g : (0, +\infty) \to R$, $g\left(u\right) = 1 - \frac{a_2}{u^2} - \frac{a_1}{u^3} - \frac{a_0}{u^4}$ is an increasing function. It results that t is the only positive root of equation $f\left(u\right) = 0$.

It result that if exists a unique continue function $u : [s_1, s_2] \to R$ such that $f\left(u\left(s\right)\right) = 0$, $\left(\forall\right) s \in [s_1, s_2]$. From implicite Theorem it follows that u is derivable on interval $\left(s_1, s_2\right)$, $u : [s_1, s_2] \to R$ which verify the condition:

$$\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)^2=4\left(\frac{2R-r}{2R}+2\sqrt{\frac{r}{4R}}u(s)\right),\ \ (\forall)\, s\in[s_1,s_2] \tag{9}$$

After we derivate the equality (9) we will obtain:

$$\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)\left(u(s)\,u'(s)-\frac{s}{4Rr}\right)=\sqrt{\frac{r}{R}}u'(s)\,,(\forall)\, s\in[s_1,s_2]$$

or in an equivalent form:

$$\left(u^3(s)-\frac{s^2+r^2-8Rr}{4Rr}u(s)-\sqrt{\frac{r}{R}}\right)u'(s)=\frac{s}{4Rr}\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)$$

or

$$\left(u^3(s)-\frac{s^2+r^2-8Rr}{4Rr}u(s)-\sqrt{\frac{r}{R}}\right)u'(s)=\frac{s}{4Rr}\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right),(\forall)\, s\in[s_1,s_2]$$

From:

$$u^2(s)=\sum\frac{s-a}{a}+2\sum\sqrt{\frac{(s-a)(s-b)}{ab}}\geq\frac{s^2+r^2-8Rr}{4Rr}+6\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{abc}}=$$

$$=\frac{s^2+r^2-8Rr}{4Rr}+6\sqrt[3]{\frac{r}{4R}},(\forall)\, s\in[s_1,s_2]$$

it results that:

$$u^3(s)-\frac{s^2+r^2-8Rr}{4Rr}u(s)-\sqrt{\frac{r}{R}}=u(s)\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)-\sqrt{\frac{r}{R}}\geq\sqrt{6\sqrt[3]{\frac{r}{4R}}}\cdot 6\sqrt[3]{\frac{r}{4R}}-$$

$$-\sqrt{\frac{r}{R}}=\left(3\sqrt{6}-1\right)\sqrt{\frac{r}{R}}>0,\ \ (\forall)\, s\in[s_1,s_2]$$

and $u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}>0,\,(\forall)\, s\in[s_1,s_2]$. It results that u is an increasing function on interval $[s_1,s_2]$.

From Theorem 3 it follows that $s_1\leq s$, for $2\leq\frac{R}{r}<\sqrt{2}+1$ which implies that $u(s_1)\leq u(s)$.

Replacing the sides a_1,b_1,c_1 of the $A_1B_1C_1$ triangle from Theorem 2 we will obtain:

$$\sum\sqrt{\frac{b+c-a}{a}}\geq\sqrt{\frac{R-r-d}{r}}+2\sqrt{\frac{R+d}{R}}\text{ if } 2\leq\frac{R}{r}<\sqrt{2}+1$$

From Theorem 3 it follows that $s_3\leq s$ if $\frac{R}{r}\geq\sqrt{2}+1$ which implies that $u(s_3)\leq u(s)$

By replacing the sides a_3, b_3, c_3 from Lemma 3 it follows that:

$$\sum\sqrt{\frac{b+c-a}{a}}\geq\sqrt{\frac{2R^2-2\sqrt{R^2-2Rr-r^2}}{R+r+\sqrt{R^2-2Rr-r^2}}}+\sqrt{\frac{2R+2\sqrt{R^2-2Rr-r^2}}{R+r-\sqrt{R^2-2Rr-r^2}}}+\sqrt{\frac{r}{R}}\text{ if }\frac{R}{r}\geq\sqrt{2}+1$$

□

Lemma 9. *In any triangle ABC is true the following inequality:*

$$\sqrt{\frac{2R-2\sqrt{R^2-2Rr-r^2}}{R+r+\sqrt{R^2-2Rr-r^2}}}+\sqrt{\frac{2R+2\sqrt{R^2-2Rr-r^2}}{R+r-\sqrt{R^2-2Rr-r^2}}}+\sqrt{\frac{r}{R}}\geq 3\textit{ if }\frac{R}{r}\geq\sqrt{2}+1 \tag{10}$$

Proof. We denote $d_2=\sqrt{x^2-2x-1}$. By squaring the inequality (10) we will obtain:

$$2\sqrt{\frac{(2x-2d_2)(2x+2d_2)}{(x+1+d_2)(x+1-d_2)}}+\frac{(2x-2d_2)(x+1-d_2)+(2x+2d_2)(x+1+d_2)}{(x+1+d_2)(x+1-d_2)}\geq\left(3-\frac{1}{\sqrt{x}}\right)^2$$

or

$$2\sqrt{\frac{4\left(x^2-x^2+2x+1\right)}{x^2+2x+1-x^2+2x+1}}+$$

$$+\frac{2x^2+2x-2xd_2-2xd_2-2d_2+2x^2-4x-2+2x^2+2x+2xd_2+2xd_2+2d_2+2x^2-4x-2}{x^2+2x+1-x^2+2x+1}\geq$$

$$\geq 9+\frac{1}{x}-\frac{6}{\sqrt{x}}$$

or

$$\frac{8x^2-4x-4}{4x+2}+2\sqrt{2}\geq 9+\frac{1}{x}-\frac{6}{\sqrt{x}}$$

or

$$2x-2+2\sqrt{2}\geq 9+\frac{1}{x}-\frac{6}{\sqrt{x}}$$

or

$$2x-11+2\sqrt{2}\geq\frac{1}{x}-\frac{6}{\sqrt{x}}$$

or

$$2x^2+\left(2\sqrt{2}-11\right)x\geq 1-6\sqrt{x}$$

or

$$2x^2+\left(2\sqrt{2}-11\right)x+6\sqrt{x}-1\geq 0$$

We consider the function $f:\left[\sqrt{2}+1,+\infty\right)\to R$

$$f(x)=2x^2+\left(2\sqrt{2}-11\right)x+6\sqrt{x}-1$$

with the derivate

$$f'(x)=4x+2\sqrt{2}-11+\frac{3}{\sqrt{x}}=4\left(x-\sqrt{2}-1\right)+6\sqrt{2}-7+\frac{3}{\sqrt{x}}\geq 0$$

It results that f is an increasing function on interval $\left[\sqrt{2}+1,+\infty\right)$ which implies that $f(x)>f\left(\sqrt{2}+1\right)$.

After performing some calculation we obtain $f\left(\sqrt{2}+1\right)>0$. □

Lemma 10. *In any triangle ABC is true the following inequality:*

$$\sqrt{\frac{R-r-d}{r}}+2\sqrt{\frac{R+d}{R}}\geq 3,\ \textit{if}\ 2\leq\frac{R}{r}\leq 8 \tag{11}$$

Proof. We denote $\frac{R}{r}=x, d_x=\frac{\sqrt{R(R-2r)}}{r}=\sqrt{x(x-2)}$. The inequality (11) may be written as:

$$\sqrt{x-1-d_x}+2\sqrt{\frac{x+d_x}{x}}\geq 3$$

By squaring we will obtain:

$$\frac{4x+4d_x}{x}\geq 9+x-1-d_x-6\sqrt{x-1-d_x}$$

or

$$6\sqrt{x-1-d_x}\geq 8-d_x+x-\frac{4x+4d_x}{x}$$

or

$$6\sqrt{x-1-d_x}\geq\frac{4x-xd_x+x^2-4d_x}{x}$$

or

$$6\sqrt{x-1-d_x}\geq\frac{(x+4)(x-d_x)}{x}$$

or

$$36x^2(x-1-d_x)\geq\left(x^2+8x+16\right)2(x-d_x-1)x$$

or

$$2x(x-d_x-1)\left(18x-x^2-8x-16\right)\geq 0 \text{ and as } x-d_x-1>0$$

It will be sufficient to prove that:

$$x^2 - 10x + 16 \leq 0 \text{ or } (x-2)(x-8) \leq 0 \text{ or } x \leq 8$$

□

Theorem 11. *(The inequality Rădulescu-Maftei) In any acute triangle is true the following inequality:*

$$\sqrt{\frac{b+c-a}{a}} + \sqrt{\frac{c+a-b}{b}} + \sqrt{\frac{a+b-c}{c}} \geq 3$$

Proof. It results from Theorem 4, Lemma 5 and 6. □

Theorem 12. *In any triangle ABC with $2 \leq \frac{R}{r} \leq 8$ is true the following inequality:*

$$\sqrt{\frac{b+c-a}{a}} + \sqrt{\frac{c+a-b}{b}} + \sqrt{\frac{a+b-c}{c}} \geq 3$$

Proof. According with the proof of Theorem 4 it follows that $u : [s_1, s_2] \to R$ is an increasing function. But $s_1 \leq s$. It results that $u(s) \geq u(s_1)$ or

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{R-r-d}{r}} + 2\sqrt{\frac{R+d}{R}} \geq 3$$

according with Lemma 6. □

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Mihály Bencze: Str. Hărmanului 6, 505600 Săcele-Négyfalu, Jud. Braşov, Romania
E-mail address: benczemihaly@yahoo.com

Marius Drăgan: 61311 bd. Timişoara Nr. 35, Bl. 0D6, Sc. E, et. 7, Ap. 176, Sect. 6, Bucureşti, Romania
E-mail address: marius.dragan2005@yahoo.com