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Side Disks of a Spherical Great Polygon

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Abstract. Take a circle and mark $n \in \mathbb{N}$ points on it designated as vertices. For any arc segment between two consecutive vertices which does not pass through any other vertex, there is a disk centered at its midpoint and has its end points on the boundary. We analyze intersection behavior of these disks and show that the number of disjoint pairs among them is between $\frac{(n-2)(n-3)}{2}$ and $\frac{n(n-3)}{2}$ and their intersection graph is a subgraph of a triangulation of a convex n-gon.

1. Introduction

An *intersection graph* of a set of figures is a graph such that there is a unique vertex associated to each figure, and two vertices are adjacent if and only if the corresponding figures are intersecting. Huemer and Perez-Lantero [4] showed that the intersection graph of a set of disks with the sides of a convex n-gon as their diameters (which are called *side disks*) is planar (see Theorem 4 in [4]). This result has a direct combinatorial consequence: the number of disjoint pairs among these disks is at least $\frac{(n-3)(n-4)}{2}$ which follows from the fact that every planar graph with n vertices has at most $3(n-2)$ edges (see Corollary 11.1(b) in [3]).

We believe that the problem of analyzing intersection patterns of side disks is of considerable interest because of the geometrical challenges resulting from its unusual conclusion, i.e. it reflects on disjointness of geometrical figures. In Euclidean geometry a search for new results by replacing line segments with conic sections is often rewarding as illustrated in the following well known results: Pappus's hexagon theorem vs. Pascal's theorem, and Ceva's theorem vs. Haruki's theorem (see Chap. 6 in [1]). Accordingly, in Sect. 2 instead of a convex n-gon we consider a circle partitioned into $n \in \mathbb{N}$ arc segments. The concept of side disk naturally extends to this setting: for each arc segment there is a unique disk centered at its midpoint and is having its two end points on its boundary. When $n = 5$ the resulting configuration already appears in Miquel's five circles theorem (see Chap. 5 in [1]). In Theorem 3 we show that for $n \geq 3$ there are at least $\frac{(n-2)(n-3)}{2}$ and at most $\frac{n(n-3)}{2}$ disjoint pairs of side disks for the partitioned circle with n arc segments. We also verify that these bounds are tight for all $n \geq 3$ and the intersection graph of these disks is a subgraph of a triangulation of a convex polygon (see Theorem 4).

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Throughout this paper we use the following conventions. For any points X, Y and Z in the plane the line passing through X, Y is denoted as XY-line, their connecting line segment is denoted as XY, and $|XY|$ is its length. $\measuredangle XYZ$ is the angle between XY and YZ measured in the clockwise direction. For any disk ω, its boundary circle is denoted as $\partial(\omega)$ and when there is no ambiguity we identify a given disk (or circle) with its center X and call it X-disk (or X-circle), etc. For any plane regions ω and τ, $(\omega \cap \tau)$ is the region in their intersection, and $\omega \subset \tau$ means the former is included (strictly) in the latter, i.e. every point in ω is in τ but not vice versa. For a point X and region τ, $X \in \tau$ means X is located in τ and $X \notin \tau$ means the opposite.

2. The main results

Let C_n be a circle partitioned into $n \in \mathbb{N}$ arc segments by marking n points on it. We identify each marked point as vertex and each arc segment between two consecutive vertices which does not pass through any other vertex as a side. Then, C_n is a spherical polygon with vertices at a great circle and we call it as *spherical great polygon*; for more on spherical polygons see e.g. Chap. 6.4 in [2]. The case where each side has the same length is denoted as $C_n^\star$. For each side of C_n, there is a unique disk centered at its midpoint and is having its two end points on the boundary. This is the *side disk* of that side and two side disks are *neighbouring* if their corresponding sides are adjacent.

Notice that each side of C_n divides its disk into two parts, one of which intersects with the region enclosed by C_n. We call this as *inner part* and the other as *outer*, and as a convention we include the corresponding arc of C_n to the inner part of the side disk, but not to the outer. Then, convexity implies that outer parts of two side disks of C_n do not intersect. We shall prove two lemmas.

Lemma 1. *Let ω be a given disk and A, B, C be points on $\partial(\omega)$ such that AC-arc is a segment of AB-arc. If ω_1 and ω_2 are the side disks of AB-arc and AC-arc, respectively, then $(\omega \cap \omega_2) \subset (\omega \cap \omega_1)$ and any point in $(\omega \cap \omega_2)$ except A is in the interior of ω_1.*

Proof. Let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Since AC-arc is contained in AB-arc, O_2 must be on the AO_1-arc not passing through B (see Fig. 1). Since O_1 is the mid-point of AB-arc, AO_1-arc is always less than a half of $\partial(\omega)$. Thus, $\measuredangle O_1O_2A > \frac{\pi}{2}$ and $\triangle AO_2O_1$ is an obtuse triangle with $|AO_1| > |AO_2|$. Let O_3 be the point on AO_1 with $|AO_3| = |AO_2|$, and ω_3 be the disk centered at O_3 and is having A on its boundary (see the dashed disk in Fig. 1). Since A, O_3 and O_1 are collinear and $|AO_1| > |AO_3|$, we have $\omega_3 \subset \omega_1$ and $A = \partial(\omega_3) \cap \partial(\omega_1)$. Then, we can conclude that $(\omega \cap \omega_3) \subset (\omega \cap \omega_1)$, and the only point in $(\omega \cap \omega_3)$ which is on $\partial(\omega_1)$ is A. On the other hand, ω_2 is a rotation of ω_3 around A in the direction to move its center from an interior point of ω, O_3, to a boundary point, O_2. Thus, we must have $(\omega \cap \omega_2) \subset (\omega \cap \omega_3)$, which implies $(\omega \cap \omega_2) \subset (\omega \cap \omega_3) \subset (\omega \cap \omega_1)$. Finally, from our proof its clear that any point in $(\omega \cap \omega_2)$ except A must be in the interior of ω_1. $\square$

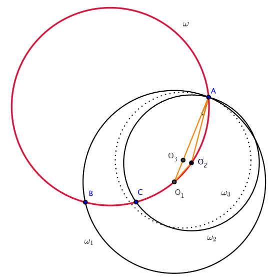

Figure 1. Illustration for Lemma 1

Lemma 2. *Let ω be a given disk and A, B, C, D be four points marked subsequently on $\partial(\omega)$. Further let ω_{ab} be the side disk corresponding to the AB-arc, and let ω_{bc}, ω_{cd} and ω_{da} be defined analogously (see Fig. 2). Let X be the intersection point of $\partial(\omega_{da})$ and $\partial(\omega_{cd})$, other than D; and Y, Z and T be defined analogously for the pairs $\partial(\omega_{cd})$ and $\partial(\omega_{bc})$, $\partial(\omega_{bc})$ and $\partial(\omega_{ab})$, and $\partial(\omega_{ab})$ and $\partial(\omega_{da})$, respectively. Then,*

(a) $X, Y \notin \omega_{ab}$, $Y, Z \notin \omega_{da}$, $Z, T \notin \omega_{cd}$ *and* $X, T \notin \omega_{bc}$; *and*
(b) *Quadrilateral $XYZT$ is a rectangle.*

Proof. To prove Lemma 2 (a), it suffices to show that $Z \notin \omega_{cd}$ as a similar argument applies to the others. Consider Fig. 2 and let the dashed disk ω_{bd} be the disk corresponding to BD-arc. It is well known, and can easily be proven that $\partial(\omega_{bd})$

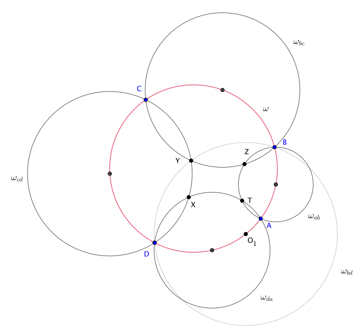

Figure 2. Illustration for Lemma 2 (a)

passes through Y, which is the incenter of $\triangle BCD$ (see below). Notice that all conditions of Lemma 1 are met for ω, ω_{bd} and ω_{ab}. Thus, $Z \in (\omega \cap \omega_{ab})$ must be located in the interior of ω_{bd}. But then $Z \notin \omega_{cd}$ as the only point which is in $(\omega_{cd} \cap \omega_{bd} \cap \omega_{bc})$ is Y, and Y and Z are distinct points as $Y \in \partial(\omega_{bd})$ while $Z \notin \partial(\omega_{bd})$. This proves Lemma 2 (a).

Let H, G, F and W be the centers of ω_{ab}, ω_{bc}, ω_{cd} and ω_{da}, respectively. We claim that X, Y, Z and T are the incenters of $\triangle ADC$, $\triangle DCB$, $\triangle CBA$ and $\triangle BAD$, respectively (see Fig. 3). Notice that since F and W are the centers of two circles intersecting at D and X, FW is a perpendicular bisector of DX and $\measuredangle XFD = 2\measuredangle WFD$. Since W is the midpoint of AD-arc, we also have $\measuredangle WFD = \frac{1}{2}\measuredangle AFD$, which implies $\measuredangle XFD = \measuredangle AFD$. Thus, points F, X and A are collinear. Since F is the midpoint of DC-arc, $\measuredangle DAF = \measuredangle FAC$, hence AF is a bisector of $\angle DAC$. Similar argument shows that W, X and C are collinear

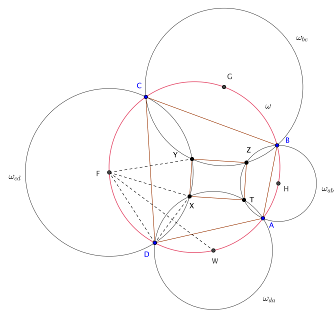

Figure 3. Illustration for Lemma 2 (b)

and CW is a bisector of $\angle DCA$. Thus, X is the incenter of $\triangle ADC$. By the same token, we may conclude that Y, Z and T are the incenters of $\triangle DCB$, $\triangle CBA$ and $\triangle BAD$, respectively. Then, the result in Lemma 2 (b) follows from Problem 6.13 in [5]. □

Remark. From Lemma 1 it follows that the inner part of a side disk of C_n is always contained in C_n. Since outer parts of side disks of C_n are disjoint, this implies that *two side disks of C_n with $n \geq 2$ intersect if and only if they intersect in the region enclosed by C_n.* To our knowledge, the only widely known result directly related to Lemma 2 is Miquel's four circles theorem which states that when ω_{ab}, ω_{bc}, ω_{cd} and ω_{da} are not necessarily centered on $\partial(\omega)$, X, Y, Z, T are concyclic (see [7]; p.151). The result used in the last step of proving Lemma 2 (b) is often referred to as the Japanese theorem (see [6]).

For C_n, let $d(C_n)$ be the number of disjoint pairs among its side disks. Our main result is as follows.

Theorem 3. *For $n \geq 3$, $\frac{(n-2)(n-3)}{2} \leq d(C_n) \leq \frac{n(n-3)}{2}$.*

Proof. Since $d(C_3) = 0$, as all side disks are neighbouring, we assume $n \geq 4$. We prove the lefthand inequality in three steps.

STEP 1: Let us prove that $1 \leq d(C_4)$.

Let A, B, C, D be the points marked on C_4 and F, G, H, W be the centers of its four side disks. Further let X, Y, Z, T be points other than A, B, C, D in which pairs of neighbouring side disks intersect by their boundaries (see Fig. 4). By Lemma 2 (b), we know that $XYZT$ is a rectangle. Let $E = XZ \cap YT$, i.e. the intersection of the diagonals of $XYZT$. We claim that $E = WG \cap FH$. Since

$|FY| = |FX|$, $|HZ| = |HT|$ and $XYZT$ is a rectangle, points F, H and the midpoints of the sides XY and ZT are collinear. Similarly, G, W and the midpoints of ZY and TX are collinear. Thus, FH and GW intersect in a point where two bimedians of $XYZT$ intersect, which must be E. This proves our claim.

Since $\measuredangle XEY + \measuredangle YEZ = \pi$ one of these two angles (summands) must be at most $\frac{\pi}{2}$, and without loss of generality we may assume that $\measuredangle XEY \leq \frac{\pi}{2}$. Then, we claim that the side disks centered at F and H are disjoint. To see this, it suffices to prove that E is located outside of these side disks, as then we have $|FE| > r_F$ and $|EH| > r_H$, hence, $|FH| = |FE| + |EH| > r_F + r_H$, where r_F and r_H are the radii of the disks to be shown as disjoint. Let us then prove that $E \notin F$-disk and a similar argument shows that $E \notin H$-disk. By Lemma 2 (a), XY-line separates F and rectangle $XYZT$. Since E is an interior point of $XYZT$, we can conclude that XY-line strictly separates E and F.

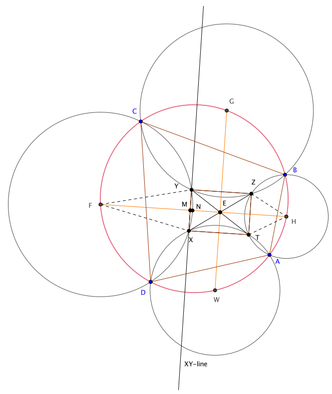

Figure 4. Side disks of C_4

Let M be the midpoint of XY and $N = \partial(F\text{-disk}) \cap FH$. It is clear that N and E lie on the same half-plane with respect to XY-line, as both are strictly separated from F by the line. We know that FH passes through the midpoints of XY and ZT, thus it must be orthogonal to XY. Recall that, both E and N lie on FH. Then, $\measuredangle XEY \leq \frac{\pi}{2}$ together with the observation that $\measuredangle XNY > \frac{\pi}{2}$ imply that $|ME| > |MN|$.[1] Then, $|FE| = |FM| + |ME| > |FM| + |MN| = r_F$. Thus, E is outside of F-disk. This proves our last claim and completes STEP 1.

STEP 2: Consider C_n and its side disks, labeled as $\omega_1, ..., \omega_n$ in the clockwise direction. For any $i, j = 1, 2, ..., n$, let (ω_i, ω_j) be the set of disks strictly between ω_i and ω_j, in the clockwise direction. We shall prove that if ω_i and ω_j intersect, then any disk in (ω_i, ω_j) is disjoint from any one in (ω_j, ω_i).

We can assume that ω_i and ω_j are non-neighbouring as the result is trivial otherwise. Let ω_i, ω_j be side disks of AB-arc and CD-arc, respectively. Then, AB-arc

[1] Take the circle centered at F. It is clear that XY is strictly shorter than its diameter. Then, for any $N^\star$ lying on the minor arc connecting X and Y, we have $\measuredangle XN^\star Y > \frac{\pi}{2}$.

and CD-arc are disjoint and we can also assume that A, B, C, D are located subsequently in the clockwise order. Let ω_{bc} and ω_{da} be the side disks of BC-arc and DA-arc, respectively (see Fig. 5). Then by STEP 1 there must be a disjoint fair among ω_i, ω_j, ω_{bc} and ω_{da}. But since the former two intersect, it must the latter two which are disjoint. By Lemma 1, we know that when restricted to the region enclosed by C_n, ω_{bc} contains all disks in (ω_i, ω_j), and similarly, ω_{da} contains all disks in (ω_j, ω_i). This implies that, none of the disks in (ω_i, ω_j) intersects with a

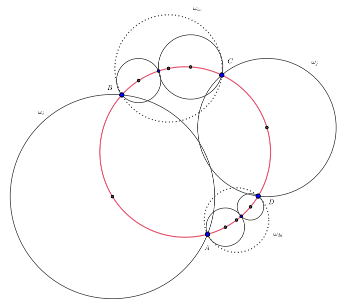

Figure 5. ω_i and ω_j intersect

disk in (ω_j, ω_i) in the region enclosed by C_n. But since two side disks intersect only in that region, we can conclude that any disk in (ω_i, ω_j) is disjoint from any one in (ω_j, ω_i). This completes STEP 2.

STEP 3: Let us prove that for $n \geq 4$, $\frac{(n-2)(n-3)}{2} \leq d(C_n)$.

Take P_n, a convex n-gon, and label its vertices with the side disks of C_n such that two disks of C_n are neighbouring if and only if their associated vertices in P_n are adjacent. Draw all $\frac{n(n-1)}{2} - n$ diagonals of P_n, and colour them with

- Red if the side disks corresponding to the end vertices intersect, and
- Blue if otherwise.

By STEP 2 we know that two red diagonals never cross in P_n. The maximal set of non-crossing diagonals of P_n gives a triangulation of it, and every triangulation involves $n-3$ diagonals (see Theorem 1.8 in [2]). Thus, the number of red diagonals is at most $n-3$, and the number of blue diagonals is at least $\frac{n(n-1)}{2} - n - (n-3) = \frac{(n-2)(n-3)}{2}$. This immediately implies that $\frac{(n-2)(n-3)}{2} \leq d(C_n)$, and completes STEP 3. The lefthand inequality in Theorem 3 is proved. Finally, since two neighbouring disks are never disjoint we have $d(C_n) \leq \frac{n(n-3)}{2}$. $\square$

Remark. It is easy to show that the upper bound in Theorem 3 is attained on $C_n^\star$, i.e. it is tight. Let $C_n^\triangle$ be a spherical great n-gon such that one of its side disks intersects with all the others, and any two of the other side disks intersect only if they are neighbouring. It is easy to show that this construction is well defined and $d(C_n^\triangle) = \frac{(n-2)(n-3)}{2}$. Thus, the lower bound is also tight for $n \geq 3$.

We can now characterize the intersection graph of side disks of C_n. Recall that a planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same (exterior) face (see Chap. 11 in [3]).

Theorem 4. *The intersection graph of side disks of C_n for $n \geq 3$ is a subgraph of a triangulation of a convex n-gon. In particular, it is outerplanar.*

Proof. Let $G(C_n)$ be the intersection graph. The result is obvious when $n = 3$. For $n \geq 4$, in STEP 2 of proof of Theorem 3 we showed that $G(C_n)$ can be drawn with no crossing edges. So, it is a subgraph of triangulation of the convex polygon with vertices at the centers of the side disks, hence outerplanar. $\square$

References

[1] M. Chamberland, *Single Digits: In Praise of Small Numbers*, Princeton University Press, 2015.

[2] S. L. Devadoss and J. O'Rourke, *Discrete and Computational Geometry*, Princeton University Press, 2011.

[3] F. Harary, *Graph Theory*, Addison-Wesley, MA, 1969.

[4] C. Huemer and P. Pérez-Lantero, The intersection graph of the disks with diameters the sides of a convex n-gon, (2016), Online: arXiv:1410.4126v3

[5] V. V. Prasolov, *Problems in Plane and Solid Geometry: Vol. 1 Plane geometry*, 3rd eds., 2001. Translated and edited by D. Leites and online at: `http://students.imsa.edu/~tliu/Math/planegeo.pdf`

[6] W. Reyes, An application of Thébault's theorem, *Forum Geom.*, 2 (2002), 183–185.

[7] D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, Penguin Books, New York, 1991.

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