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Parallelograms Inscribed in Convex Quadrangles

Nikolaos Dergiades

Abstract. In this article we construct the inscription in a quadrangle of a parallelogram similar to a given one and we study the parallelograms inscribed in quadrangles with special emphasis on determining the classes of quadrangles which allow the inscription of infinite many parallelograms of a given similarity type. As special cases we obtain the characterization of quadrangles allowing the inscription of infinite many rhombi of a fixed similarity type, of infinite many rectangles of a fixed similarity type and of quadrangles allowing the inscription of infinite many squares.

1. A simple case of inscribed parallelograms

In an arbitrary convex quadrilateral $q = ABCD$ we can easily inscribe infinite many parallelograms with sides parallel to the diagonals $\{AC, BD\}$ of the quadrilateral of reference q (See Figure 1). The vertex A_1 of such a parallelogram is

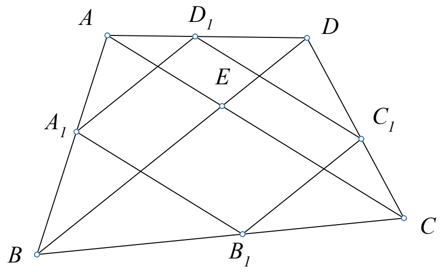

Figure 1. Parallelograms with sides parallel to the diagonals

selected on the side AB. Then, drawing a parallel to the diagonal AC we define its intersection point B_1 with BC. Drawing further from B_1 a parallel to BD we define its intersection C_1 with CD and continuing this way, we obtain, using Thales theorem, an inscribed parallelogram $A_1B_1C_1D_1$ with sides parallel to the diagonals of q.

It is easy to see that among these parallelograms there exist one that is a rhombus satisfying the following property:

Lemma 1. *The rhombus $A_1B_1C_1D_1$, inscribed in the quadrangle $q = ABCD$, with sides parallel to the diagonals of q is constructible with ruler and compasses.*

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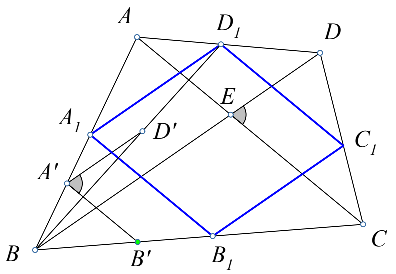

Figure 2. Inscribing a rhombus $A_1B_1C_1D_1$ in the quadrangle $ABCD$

Proof. In fact, from an arbitrary point B' on the side BC, draw the parallel to CA, that meets BA at the point A' (See Figure 2). Rotate then B' about A' by the angle of the diagonals $\widehat{CED}$ to get the point D'. The line BD' meets AD at the point D_1, from which with parallels to the diagonals of q we construct the other vertices of $A_1B_1C_1D_1$, seen to be a rhombus by applying the Thales theorem

$$\frac{A_1D_1}{A'D'} = \frac{BA_1}{BA'} = \frac{A_1B_1}{A'B'}.$$

□

2. The inscription in a quadrangle of a parallelogram

Given a convex quadrangle $q = ABCD$ with positive orientation and a parallelogram $A_0B_0C_0D_0$ with $\widehat{A_0C_0B_0} = \alpha$, $\widehat{B_0A_0C_0} = \gamma$ and $C_0B_0A_0 = \beta$, it is very interesting the problem of inscribing in q a parallelogram $A_1B_1C_1D_1$ similar to $A_0B_0C_0D_0$.

(1) $\beta = \pi/2$, that is $A_0B_0C_0D_0$ is a rectangle.

(a) $\alpha \neq \gamma$.

The rotation of line AD about A towards B by the angle α gives the line L_1. The rotation of line BC, about B towards A by the angle γ gives the line L_2. The perpendicular from A to line L_2 meets the line BC at A' and the parallel from A' to line L_2 meets the line L_1 at the point A'' (See Figure 3). Similarly the perpendicular from B to line L_1 meets the line AD at B' and the parallel from B' to line L_1 meets the line L_2 at the point B''. Let C_1 be the intersection of line $A''B''$ with the line CD. The parallels from C_1 to $\{L_1, L_2\}$ meet the lines $\{AD, BC\}$ respectively at the points $\{D_2, B_2\}$ and the perpendicular from B_2 to L_2 meets AB at the point A_1. From Thales theorem we have

$$\frac{AA_1}{A_1B} = \frac{A'B_2}{B_2B} = \frac{A''C_1}{C_1B''} = \frac{AD_2}{D_2B'},$$

which means that A_1D_2 is parallel to BB', which is perpendicular to L_1. Hence $\widehat{A_1D_2C_1} = \widehat{A_1B_2C_1} = \pi/2$ and the circle on diameter A_1C_1 passes through $\{B_2, D_2\}$ and meets again the lines $\{BC, AD\}$

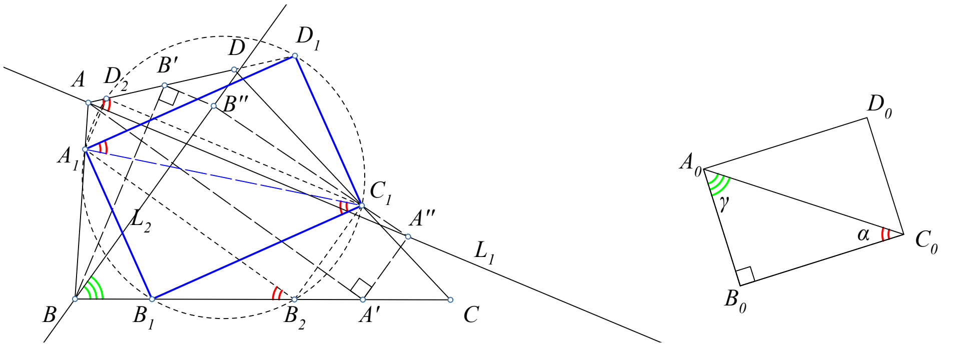

Figure 3. The inscription in a quadrangle of a rectangle

respectively at points $\{B_1, D_1\}$. Since A_1B_2 is perpendicular to L_2, the angle $\widehat{BB_2A_1} = \alpha$. Since $\widehat{B_1C_1A_1} = \widehat{B_1B_2A_1} = \alpha$, $\widehat{C_1A_1D_1} = \widehat{C_1D_2D_1} = \alpha$ and $\widehat{C_1B_1A_1} = \widehat{A_1D_1C_1} = \pi/2$, the quadrangle $A_1B_1C_1D_1$ is a rectangle similar to $A_0B_0C_0D_0$.

Note: If the line $A''B''$ is parallel to CD then the problem has no solution. If the line $A''B''$ coincides with line CD, then the problem has infinite many solutions. We'll see these cases later on.

(b) $\alpha = \gamma = \pi/4$, that is $A_0B_0C_0D_0$ is a square.
The construction is obviously similar to the previous one.

(2) $\beta \neq \pi/2$.

(a) $\alpha \neq \gamma$, that is $A_0B_0C_0D_0$ is not a special parallelogram.
The construction differs slightly from the previous one.
The rotation of line AD about A towards B by the angle α gives the line L_1. The rotation of line BC, about B towards A by the angle γ gives the line L_2. The rotation of the parallel line from A to BC, about A towards B by angle α gives the line AA', where A' is on BC, such that $\widehat{BA'A} = \alpha$. Similarly the rotation of the parallel line from B to AD, about B towards A by angle γ gives the line BB', where B' is on AD, such that $\widehat{AB'B} = \gamma$. The parallel from A' to L_2 meets L_1 at the point A'' and the parallel from B' to L_1 meets L_2 at the point B'' (See Figure 4). Let C_1 be the intersection of line $A''B''$ with the line CD. The parallels from C_1 to $\{L_1, L_2\}$ meet the lines $\{AD, BC\}$ respectively at the points $\{D_2, B_2\}$ and the parallel from B_2 to AA' meets AB at the point A_1. From Thales theorem we have

$$\frac{AA_1}{A_1B} = \frac{A'B_2}{B_2B} = \frac{A''C_1}{C_1B''} = \frac{AD_2}{D_2B'},$$

which means that A_1D_2 is parallel to BB'. It is then easy to see that $\widehat{A_1D_2C_1} = \widehat{A_1B_2C_1} = \pi - \alpha - \gamma = \beta$. If B_1 is the second intersection of the circumcircle of the triangle $A_1B_2C_1$ with the

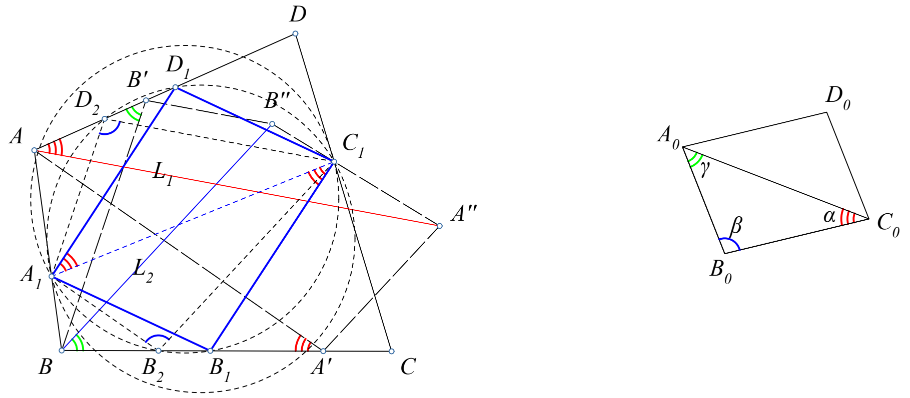

Figure 4. The inscription in a quadrangle of a general parallelogram

line BC and D_1 is the second intersection of the circumcircle of triangle $A_1D_2C_1$ with line AD, then we have $\widehat{A_1C_1B_1} = \widehat{A_1B_2B} = \widehat{AA'B} = \alpha$, $\widehat{C_1A_1D_1} = \widehat{C_1D_2D_1} = \alpha$ and $\widehat{C_1B_1A_1} = \widehat{A_1D_1C_1} = \beta$, which means that $A_1B_1C_1D_1$ is a parallelogram similar to $A_0B_0C_0D_0$.

(b) $\alpha = \gamma$, that is, $A_0B_0C_0D_0$ is a rhombus.
The construction is similar to the previous one.

3. Similar quadrangles inscribed in the quadrangle

The situation described in the preceding section changes radically, when we require from the quadrangle to have infinite many inscribed quadrangles of a *fixed similarity type*. This implies severe restrictions to the circumscribing quadrilateral, which we study in this section. To fix the ideas, we assume that the quadrangle $q_1 = A_1B_1C_1D_1$ is inscribed in the quadrangle $q = ABCD$, when the vertices $\{A_1, B_1, C_1, D_1\}$ are respectively on the sides $\{AB, BC, CD, DA\}$. Next theorem expresses a basic feature of the configuration under consideration.

Theorem 2. *If in the quadrangle $q = ABCD$, there are inscribed two directly similar quadrangles $\{q_1, q_2\}$, then the following properties hold.*

(1) *There are infinite many other quadrangles $\{q_t\}$ directly similar to q_1 and inscribed in q.*
(2) *The similarity center P of the quadrangles $\{q_1, q_2\}$ defines through its projections on the sides of q its pedal quadrangle q_0, which is the minimal similar to q_1 quadrangle.*
(3) *Every pair of the family of directly similar inscribed quadrangles $\{q_1, q_2\}$ has the same similarity center P.*

Proof. The claims are consequences of well known facts about direct similarities, an exposition of which can be found in [7, vol.II] and [5, p.74]. The center of the similarity is the Miquel point P ([3, p.79], [4, p.131]) of one the resulting, inscribed in $ABCD$, triangles, $A_1B_1C_1$ say (See Figure 5), and is identified with

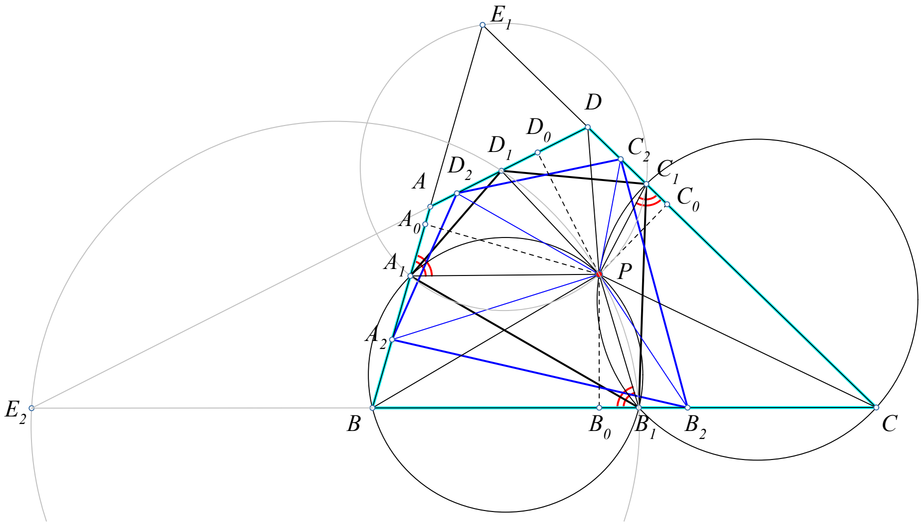

Figure 5. The similarity center P of $\{q_1 = A_1B_1C_1D_1, q_2 = A_2B_2C_2D_2\}$

the intersection P of the circles $\{(A_1BB_1), (B_1CC_1), (E_1A_1C_1)\}$. A simple angle chasing argument shows that the angles ([4, p. 142])

$$\widehat{BPC} = \widehat{A_1B_1C_1} + \widehat{A_1E_1C_1}.$$

And a similar argument shows that $\widehat{CPE_1} = \widehat{A_1C_1B_1} + \widehat{E_1BC}$. Thus point P is the same for the two quadrangles $\{q_1, q_2\}$ and the triangles $\{PA_1A_2, PB_1B_2, PC_1C_2\}$ are similar. Analogous work with the triangle E_2CD and the inscribed in it triangles $B_1C_1D_1$ and $B_2C_2D_2$, shows that P is also the similarity center for these two triangles and the triangles $\{B_1PB_2, C_1PC_2, D_1PD_2\}$ are also similar. The claims of the theorem follow easily from these remarks. $\square$

Notice that, as a parameter $t \in \mathbb{R}$ in the family of inscribed quadrangles $\{q_t\}$ we can consider the signed distance of one of its vertices, B_t say, from the corresponding projection B_0 of P. All the quadrangles q_t result by defining the similar and similarly oriented right triangles $\{PA_0A_t, PB_0B_t, PC_0C_t, PD_0D_t\}$.

Corollary 3. *A necessary and sufficient condition for the possibility, to inscribe in a quadrangle q infinite many, directly similar to each other quadrangles q_t, is the existence of a point P, whose pedal quadrangle with respect to q is similar to q_t.*

4. Similar parallelograms inscribed in a quadrangle

By theorem 2, if we have two similar parallelograms inscribed in the convex quadrangle $q = ABCD$, then we have infinite many similar to those two parallelograms, inscribed in q. We can fix the parallelogram and its pivot point P and consider the circumscribed quadrangles q_t of the parallelogram, which are all similar to the antipedal $q_1 = A_1B_1C_1D_1$ of P with respect to the parallelogram (See

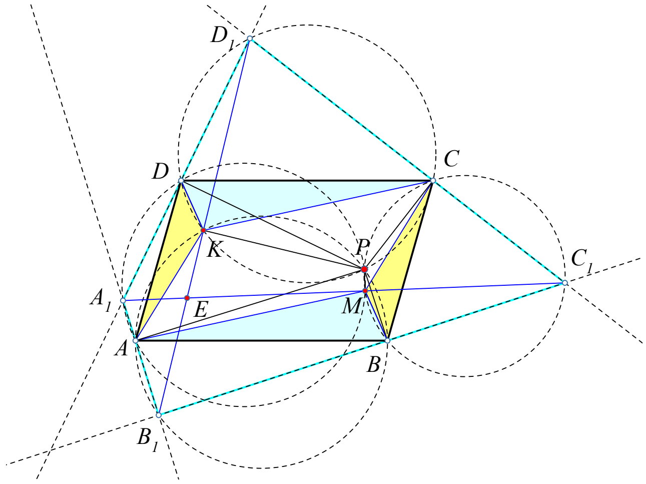

Figure 6. Angles of diagonals of antipedal with respect to a parallelogram

Figure 6). Next theorem relates the angles of the parallelogram to those of the diagonals of the antipedal.

Theorem 4. *Let $q_1 = A_1B_1C_1D_1$ be the antipedal quadrangle of a parallelogram $q = ABCD$ with respect to the point P. Then, the angles of the diagonals of q_1 are equal to the angles of the parallelogram q.*

Proof. Consider the intersections of the pairs of circles $M = (PBC) \cap (PAD)$ and $K = (PDC) \cap (PAB)$. Points $\{K, M\}$ are symmetric with respect to the center of the parallelogram and define equal triangles $\{ADK, CBM\}$ (See Figure 6). This is seen by examininig the angles of these triangles. For example, the angle $\widehat{DKA}$, by taking the adjacents to it angles, which sum to 2π, is seen to be equal to the sum

$$\widehat{DKA} = \widehat{DCP} + \widehat{PBA} = \widehat{CPB} = \widehat{CMB}.$$

Analogously is seen that $\widehat{DKC}$ is equal to $\widehat{AMB}$, so that the points are symmetric as stated. It is easily seen also that the diagonals of q_1 pass respectively through $\{K, M\}$ and make there respectively with $\{PK, PM\}$ right angles, so that the quadrangle $PKEM$, formed by their intersection, is cyclic. But the angle of the diagonals

$$\widehat{KEM} = \pi - \widehat{KPM} \text{ and } \widehat{KPM} = 2\pi - \widehat{KPC} - \widehat{MPC} = \widehat{KDC} + \widehat{MBC} = \widehat{ABC}.$$

□

Theorem 5. *With the notation and conventions of the preceding theorem, the pivot point P of $q_1 = A_1B_1C_1D_1$ is viewing each side of q_1 by an angle, which is the sum of the angles viewing that side from the other two vertices.*

Proof. This follows by splitting the angle (See Figure 7)

$$\widehat{B_1PC_1} = \widehat{B_1PB} + \widehat{BPC_1} = \widehat{B_1AB} + \widehat{BCC_1}.$$

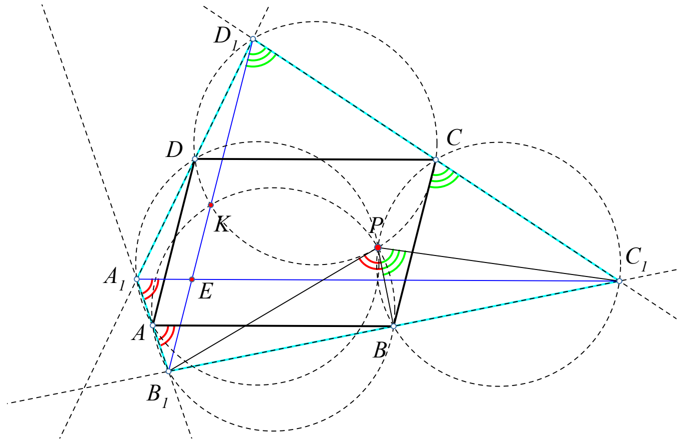

Figure 7. How the pivot of $ABCD$ is viewing the sides of $A_1B_1C_1D_1$

By the previous theorem, fixing q_1 and pivoting $ABCD$, so that its sides become parallel to the diagonals of q_1, we have that the sum in the last formula is equal to $\widehat{B_1A_1C_1} + \widehat{B_1D_1C_1}$. This proves the claim for the side B_1C_1, the proof for the other sides being completely analogous. □

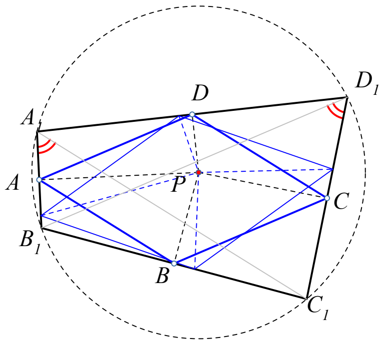

Figure 8. In cyclic quadrangles the pivoting parallelogram is that of Varignon

Corollary 6. *If $q_1 = A_1B_1C_1D_1$ is a cyclic quadrangle, then the pivoting parallelogram $q = ABCD$ of it is the one of Varignon, defined by the middles of the sides of q_1 and the pivot P is the center of the circumcircle.*

Proof. By theorem 5 the pivot P is then viewing B_1C_1 under the ange $\widehat{B_1PC_1} = 2\widehat{B_1AC_1}$, which identifies P with the center of the circumcircle (See Figure 8). □

The converse is also true and its proof is left as an exercise.

Corollary 7. *If the pivoting parallelogram inside the quadrangle q_1 is similar to the Varignon parallelogram of the middles of the sides, then the quadrangle q_1 is cyclic.*

Next corollary is also easy to show using the figure 9, and is left also as an

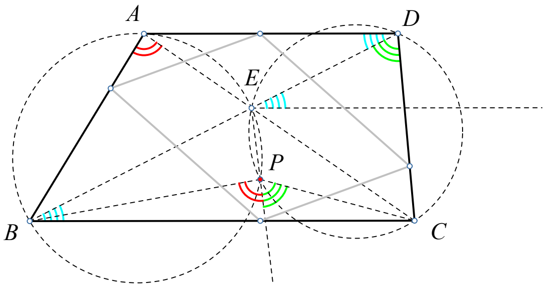

Figure 9. The pivot P of similar parallelograms inscribed in a trapezium

exercise.

Corollary 8. *The pivot P of the similar parallelograms pivoting inside a trapezium is the second intersection point of the circles $\{(EAB), (EDC)\}$ through the intersection of the diagonals $E = (AC, BD)$ and the end points of the non-parallel sides.*

In the case $q = ABCD$ is a parallelogram, the points $\{E, P\}$ in figure 9 coincide and the pivot is the center of the parallelogram.

Corollary 9. *The pivot of similar parallelograms inside a parallelogram $q = ABCD$ is the center P of the parallelogram q.*

5. Alternative approach for the pivot of similar parallelograms

It is possible to construct special parallelograms, from the family of those pivoting about a point P in the quadrangle $q = ABCD$. A particular one, of importance for the following development is given by the next theorem.

Theorem 10. *If the distances of the intersection point E of the diagonals of the quadrilateral $q = ABCD$ from the vertices are correspondingly $\{a, b, c, d\}$, then there is a parallelogram $q_1 = A_1B_1C_1D_1$ of the family of similar parallelograms pivoting inside q, such that*

$$\frac{BB_1}{B_1C} = \frac{bd}{ac}.$$

Proof. Let the parallel ε from E to BC intersect AB at B' and C' be the symmetric of B' with respect to E (See Figure 10). Let the circle $\kappa = (AB'C)$ intersect ε at B'' and the circle $\lambda = (BC'D)$ intersect the line ε at C''. Consider also the intersection of lines $F = (BC'', CB'')$. Then $B_1 = (EF, BC)$ is the required point.

For the proof consider the power of E with respect to λ : $EC' \cdot EC'' = EB \cdot ED = bd$. Similarly, the power of E with respect to κ is $EB' \cdot EB'' = EA \cdot EC = ac$. It follows that the ratio

$$\frac{BB_1}{B_1C} = \frac{C''E}{EB''} = \frac{bd/EC'}{ac/EB'} = \frac{bd}{ac}.$$

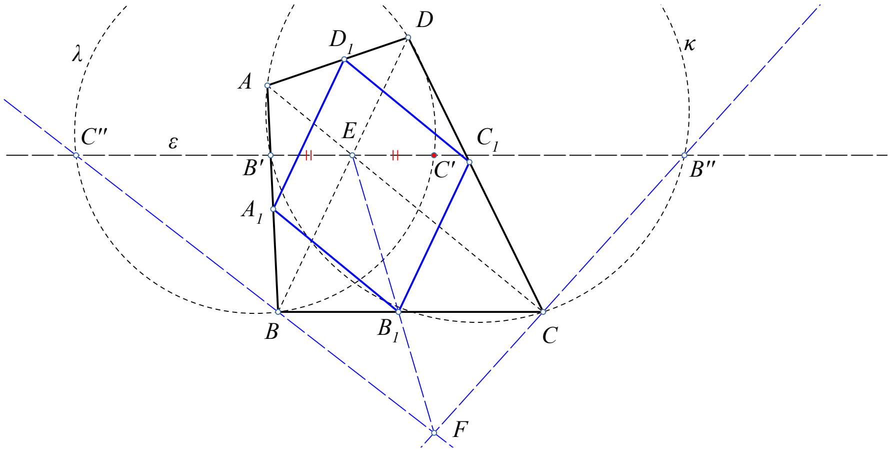

Figure 10. Special inscribed parallelogram

□

Theorem 11. *With the notation and conventions of the preceding theorem, the circles* (AA_1D_1), (BB_1A_1), (CC_1B_1) *and* (DD_1C_1) *pass through the same point* P*, which is the pivot of the similar parallelograms inscribed in* q*.*

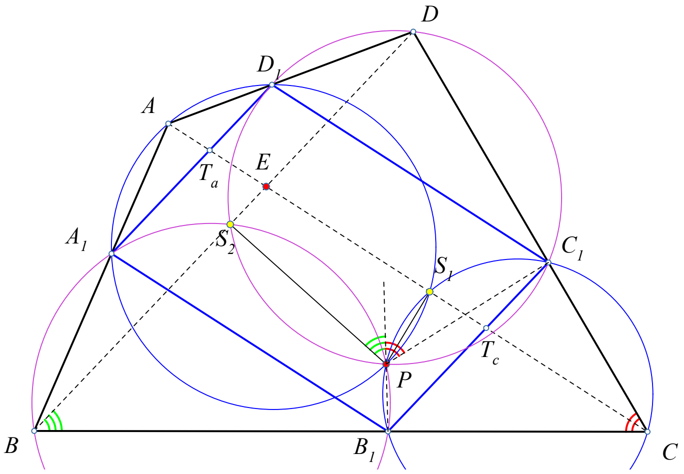

Figure 11. The pivot determined by the special inscribed parallelogram

Proof. Using $\{EA = a, EB = b, EC = c, ED = d\}$, the assumption $\frac{BB_1}{B_1C} = \frac{bd}{ac}$ and denoting the intersections of lines by $\{T_a = (A_1D_1, AC), T_c = (B_1C_1, AC)\}$ (See Figure 11), we have

$$\frac{AA_1}{AB} = \frac{B_1C}{BC} = \frac{ac}{ac+bd} = \frac{AT_a}{AE} = m \quad \Rightarrow \quad AT_a = am.$$

Similarly we get

$$A_1T_a = B_1T_c = bm, \qquad T_aD_1 = T_cC_1 = dm, \qquad T_cC = cm.$$

Let S_1 be the second intersection of the circle (AA_1D_1) with AC. Then, we have the relations

$$AT_a \cdot T_aS_1 = A_1T_a \cdot T_aD_1 \quad \Rightarrow \quad T_aS_1 = \frac{bdm}{a} \quad \Rightarrow$$

$$\begin{aligned} S_1T_c &= (AC - AT_a - T_cC) - T_aS_1 = (a+c)(1-m) - \frac{bdm}{a} \\ &= (a+c)\frac{bd}{ac+bd} - \frac{bdc}{ac+bd} = \frac{bdm}{c} \quad \Rightarrow \quad S_1T_c \cdot T_cC = B_1T_c \cdot T_cC_1. \end{aligned}$$

This means that the circle (B_1CC_1) also passes through point S_1. Similarly we show that the circles $\{(BB_1A_1), (DD_1C_1)\}$ intersect a second time at the point S_2 on the diagonal BD. Let now P be the second intersection point of the circles $\{(BB_1A_1), (CC_1B_1)\}$. Then, we have $\widehat{C_1PS_2} = \widehat{CBD} + \widehat{DCB} = \pi - \widehat{C_1DS_2}$, i.e. P is on the circle $(DS_2C_1) = (DD_1C_1)$. Similarly we prove that P is on the circle (AA_1D_1). □

6. Similar rectangles inscribed in a quadrangle

By theorem 2 the condition to have infinite many similar rectangles inscribed in a quadrangle is equivalent with the existence of a point whose pedal quadrangle is a rectangle. Next theorem supplies some basic implications for such a case.

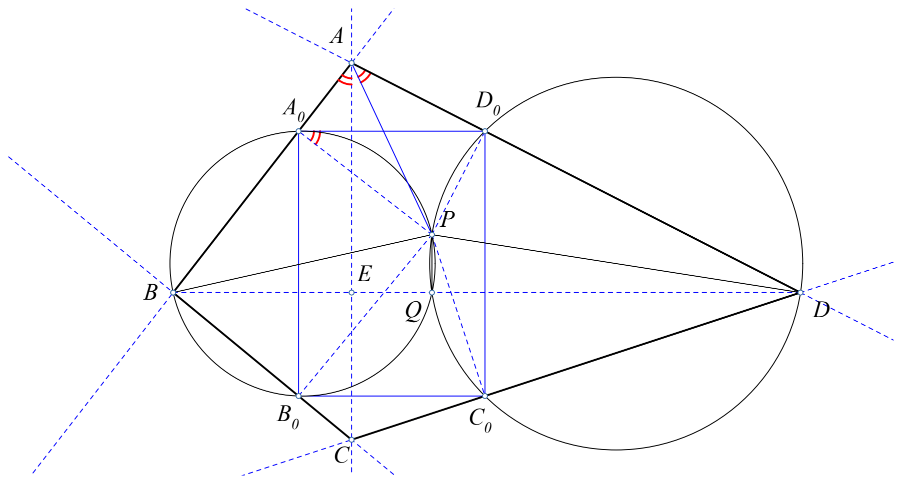

Figure 12. Quadrangle q possessing a pedal q_0, which is a rectangle

Theorem 12. *If the pedal quadrangle $q_0 = A_0B_0C_0B_0$ of a point P, relative to the quadrangle $q = ABCD$ is a rectangle, then*

(1) *The sides of q_0 are parallel to the diagonals of q, i.e. q is orthodiagonal.*
(2) *The intersection point E of the diagonals of q is isogonal conjugate to P.*

Proof. For *nr-1* consider the intersection Q of the circles $\{PC_0DD_0), (PB_0BA_0)\}$ (See Figure 12). Since $A_0B_0C_0D_0$ is a rectangle, the figure consisting of it and the two circles is symmetric with respect to the line joining the middles of the opposite sides $\{A_0B_0, C_0D_0\}$, hence line PQ is parallel to these sides too. By the inscribed quadrangles in these circles, the angles at Q are both right, so that $\{B, Q, D\}$ are collinear, hence BD is orthogonal to A_0B_0. Analogously is seen the orthogonality of AC to A_0D_0.

Nr-2 follows from *nr-1* and a simple angle chasing argument, showing that the angles $\widehat{PAD_0} = \widehat{A_0AE}$. Handling analogously the angles of the rays from the vertices of q to points $\{E, P\}$, we see indeed the isogonal property of $\{E, P\}$. $\square$

Corollary 13. *Under the conditions of the preceding theorem, the points $\{E, P\}$ are also isogonal with respect to the triangles $\{E_1BC, E_1DA, E_2AB, E_2CD\}$, for the intersections of opposite sides $\{E_1 = (AB, CD), E_2 = (AD, BC)\}$.*

7. Similar rhombi inscribed in a quadrangle

In this section we specialize from similar parallelograms to similar rhombi inscribed in a quadrangle. Next lemma establishes a property of the triangle used below.

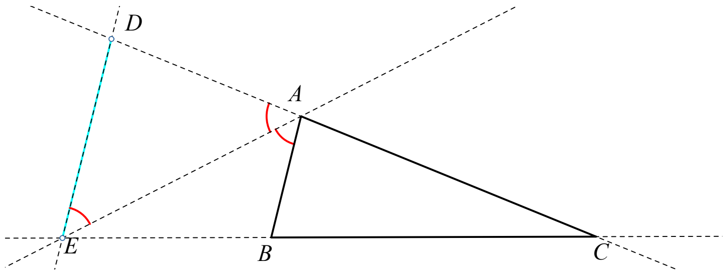

Figure 13. A property of the triangle

Lemma 14. *If AE is the external bisector of the triangle ABC with $AB < AC$, and the parallel from E to AB meets AC at D, then we have*

$$\frac{1}{AB} - \frac{1}{AC} = \frac{1}{ED}.$$

Proof. The triangle DEA is isosceles $DA = DE$. Hence we have (See Figure 13)

$$\frac{DE}{AB} = \frac{DC}{AC} = \frac{DA + AC}{AC} = \frac{DE + AC}{AC} = \frac{DE}{AC} + 1 \quad \text{or} \quad \frac{1}{AB} - \frac{1}{AC} = \frac{1}{ED}.$$

$\square$

Lemma 15. *Let $q = ABCD$ be a quadrangle whose triangle of diagonal points of the corresponding complete quadrilateral is EE_1E_2 (See Figure 14). Then, the following properties are equivalent.*

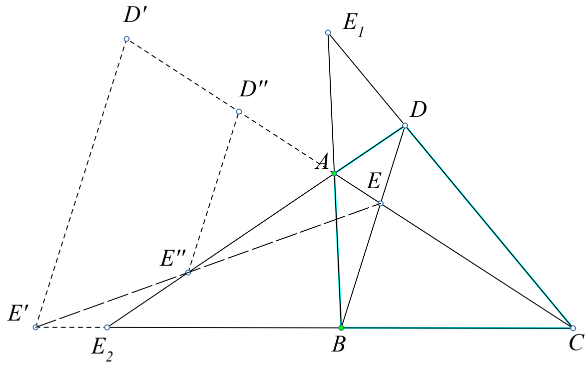

Figure 14. Bisecting the angle of the diagonals

(1) *The triangle* EE_1E_2 *has a right angle at* $E = (AC, BD)$.
(2) *The lines* $\{EE_1, EE_2\}$ *are bisectors of the angle of the diagonals* $\{AC, BD\}$ *of* q.
(3) *The harmonic means of the segments on the diagonals are equal:* $\frac{1}{EA} + \frac{1}{EC} = \frac{1}{EB} + \frac{1}{ED}$.

Proof. The equivalence of *nr-1* and *nr-2* follows from the general fact that the diagonals of a quadrilateral and the lines $\{EE_1, EE_2\}$ form a harmonic pencil ([1, p.101]). Thus, if two of the arrays of the pencil make a right angle, then they bisect the angle of the other two ([1, p.102]).

The equivalence of *nr-2* and *nr-3* results by considering the intersections $\{E', E''\}$ of the bisector of the angle $\widehat{AEB}$ respectively with $\{BC, AD\}$. The parallels from $\{E', E''\}$ to the diagonal BD intersect the other diagonal AC respectively at the points $\{D', D''\}$. Applying lemma 14 respectively to the triangles $\{EBC, AED\}$ we obtain

$$\frac{1}{EB} - \frac{1}{EC} = \frac{1}{E'D'} = \frac{1}{ED'} \qquad \text{and} \qquad \frac{1}{AE} - \frac{1}{ED} = \frac{1}{ED''}.$$

Thus, the assumed condition is equivalent to

$$\frac{1}{EA}+\frac{1}{EC} = \frac{1}{EB}+\frac{1}{ED} \quad \Leftrightarrow \quad \frac{1}{EA}-\frac{1}{ED} = \frac{1}{EB}-\frac{1}{EC} \quad \Leftrightarrow \quad \frac{1}{ED''} = \frac{1}{ED'},$$

which means that $\{E', E''\}$ must coincide with E_2. □

Theorem 16. *The necessary and sufficient condition to inscribe in a quadrangle infinite many similar rhombi, is that the bisectors of the angle of the diagonals pass through the intersections of the opposite sides of the quadrangle.*

Proof. Let again $\{EA = a, EB = b, EC = c, ED = d\}$ denote the segments to the vertices from the intersection point $E = (AC, BD)$ of the diagonals of the quadrangle $q = ABCD$ (See Figure 15). By theorem 10 we have a parallelogram $q_1 = A_1B_1C_1D_1$ of the infinite family of inscribed parallelograms, which has

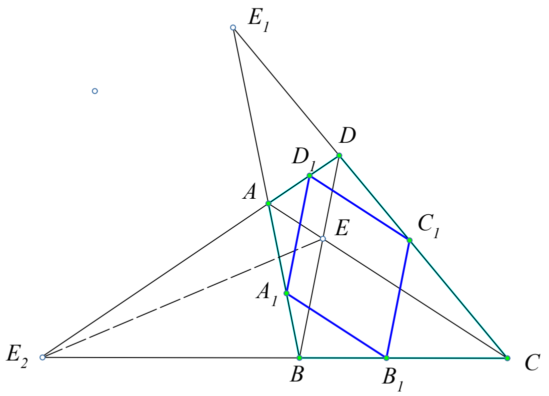

Figure 15. Condition for infinite many pivoting rhombi

sides parallel to the diagonals of q and satisfies $\frac{BB_1}{B_1C} = \frac{bd}{ac}$. For this parallelogram we obtain the relation

$$\frac{B_1C_1}{BD} = \frac{B_1C}{BC} \quad \Rightarrow \quad B_1C_1 = \frac{ac(b+d)}{ac+bd} \quad \text{and similarly} \quad A_1B_1 = \frac{bd(a+c)}{ac+bd}.$$

The parallelogram being a rhombus iff $A_1B_1 = B_1C_1$ combined with the preceding relations gives the necessary and sufficient condition ([6])

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d},$$

which, by lemma 15, is the necessary and sufficient condition for the bisectors, to pass respectively through points $\{E_1, E_2\}$. □

Theorem 17. *If a quadrangle $q = ABCD$ has infinite many inscribed rhombi, then for the rhombus $A_1B_1C_1D_1$ of this family, whose sides are parallel to the diagonals of q we have $BB_1 : B_1C = BD : AC$.*

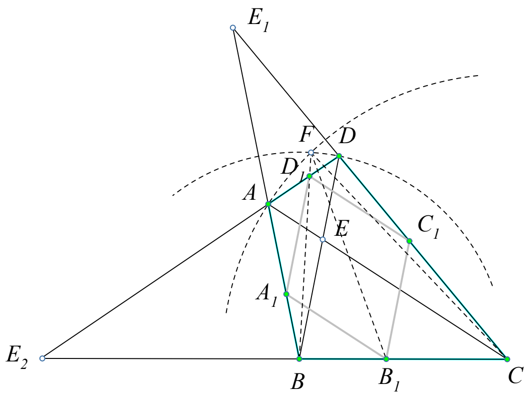

Figure 16. Constructing the pivoting rhombus with side parallel to the diagonals

Proof. By theorem 16 and lemma 15, we know that such a quadrangle q is characterized by the condition $\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d}$, hence $bd : ac = BD : AC$. Hence from the known relation for this q_1 : $BB_1 : B_1C = bd : ac$ we get $BB_1 : B_1C = BD : AC$. Thus, the point B_1 can be constructed using the bisector FB_1 of the triangle FBC, taking F with $FB = BD$ and $FC = AC$ (See Figure 16). □

8. Impossibility to inscribe similar parallelograms

From the theory exposed so far it is clear that if we are given a quadrangle and a parallelogram $\{q = ABCD, q_0 = A_0B_0C_0D_0\}$ and we want to inscribe in q infinite many parallelograms $q_t = A_tB_tC_tD_t$ similar to q_0, with A_t on AB, B_t on BC etc., then the following theorem holds.

Theorem 18. *Let E denote the intersection of the diagonals of q and $EA = a$, $EB = b$, $EC = c$, $ED = d$. Then it is impossible to inscribe in q infinite many parallelograms similar to q_0 if one of the following conditions is valid.*

(1) $A_0B_0 : B_0C_0 \neq \left(\frac{1}{a} + \frac{1}{c}\right) : \left(\frac{1}{b} + \frac{1}{d}\right)$.

(2) $\widehat{BEC} \neq \widehat{A_0B_0C_0}$.

9. Impossibility to inscribe a given parallelogram

We saw in section 2 that the inscription in a quadrangle $q = ABCD$ of a parallelogram $q_1 = A_1B_1C_1D_1$, similar to $q_0 = A_0B_0C_0D_0$ is impossible if the point $D_1 = (A''B'', CD)$ is at infinity, i.e. the line $A''B''$ is parallel to line CD. Using homogeneous barycentric coordinates with respect to the triangle ABC and considering the point D variable with coordinates (x, y, z) we can ask for the geometric locus of D, so that the line $A''B''$ is parallel to the line CD. It turns out that this locus is a conic, so that *if the point D is on that conic, then the problem has no solution*. In order to reduce the complexity, we don't give further explanations and stay in the case $\alpha = \gamma$, i.e. the case in which q_0 is a rhombus, for which we note by $\beta = \widehat{A_0B_0C_0}$. Then the following theorem is valid.

Theorem 19. *It is impossible to inscribe in q a rhombus $q_1 = A_1B_1C_1D_1$ similar to q_0, with $\widehat{A_0B_0C_0} = \beta$, when $D_1 = (A''B'', CD)$ is at inifinity, which is equivalent with the condition that $D(x, y, z)$ lies on the conic represented in the aforementioned barycentric coordinates by the equation*

$$\sin(\beta)(a^2yz + b^2zx + c^2xy) + (x + y)(y + z)S = 0,$$

where $\{a, b, c, S\}$ are the lengths of the sides and S is the double area of ABC.

Proof. It is left to the reader as a result of computer aided calculations, such as in [2]. □

In order to construct this conic we need to know 5 points on it. Since

$$C_c : \ a^2yz + b^2zx + c^2xy = 0, \quad P_{bc} : \ y + z = 0, \quad P_{ab} : \ x + y = 0,$$

are respectively the equations of the circumcircle of ABC and the lines which are parallel from $\{A, C\}$ to $\{BC, BA\}$, it is easy to see that four of these points are

$\{A, X, C, Z\}$, where $\{P_{bc}, P_{ab}\}$ meet respectively the circle C_c. A fifth special point D' on the conic is constructed as follows.

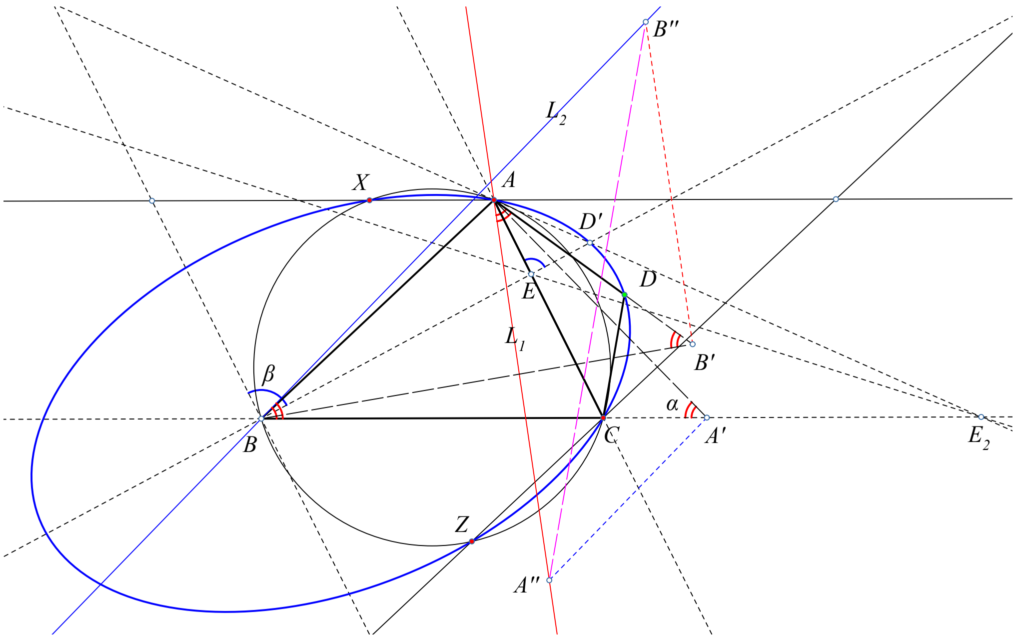

Figure 17. The conic locus of D, such that $A''B''$ is parallel to CD

The parallel from B to AC is rotated about B towards A by the angle β to give line BED', where E is the intersection with AC (See Figure 17). The bisector of the angle $\widehat{CED'}$ meets the line BC at the point E_2 and the line AE_2 meets the line BE at the required point D'. This lies on the conic, because in the quadrangle $ABCD'$ we can inscribe infinite many rhombi similar to $A_0B_0C_0D_0$. Now we consider the conic through the points $\{A, X, Z, C, D'\}$ and take on this a point D, such that the quadrangle $ABCD$ is convex. Since $\alpha = \gamma$ we have $\alpha = (\pi - \beta)/2$ and as in section 2 we construct lines $\{L_1, L_2\}$ and the line $A''B''$, that now is seen to be parallel and different from the line CD, for every point D on this conic, except for the point D', where lines $A''B''$ and CD coincide.

If the angle $\beta = \pi/2$, then the rhombus is a square, the line BD' is perpendicular to AC, the equation of the conic and the baricentric coordinates of D' are

$$a^2yz + b^2zx + c^2xy + (x + y)(y + z)S = 0,$$
$$D' = (2(a^2 + b^2 - c^2)S : -(a^2 + b^2 - c^2)(b^2 + c^2 - a^2) : 2(b^2 + c^2 - a^2)S).$$

Hence we have the following final result.

Theorem 20. *Given a convex quadrangle $q = ABCD$ and a rhombus $r_0 = A_0B_0C_0D_0$ with $\widehat{A_0B_0C_0} = \beta$, then*

(1) *If the point D is not on the conic described previously, we can inscribe in q only one rhombus similar to r_0.*

(2) *If the point D is on the conic and $D = D'$, we can inscribe in q infinite many rhombi similar to r_0.*
(3) *If the point D is on the conic and $D \neq D'$, it is impossible to inscribe in q a rhombus similar to r_0.*

References

[1] H. Eves, *A survey of Geometry*, Allyn and Bacon, Inc., Boston, 1963.
[2] F. J. García Capitán, Coordenadas Baricentricas, 2013;
http://garciacapitan.esy.es/baricentricas/.
[3] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, The Mathematical Association of America, New Library, Washington, 1995.
[4] R. A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, New York, 1960.
[5] T. Lalesco, *La Geometrie du triangle*, Librairie Vuibert, Paris, 1952.
[6] D. Laurain, Advanced Geometry post. https://groups.yahoo.com/neo/groups /AdvancedPlaneGeometry/conversations/messages/4574, 2018.
[7] I. M. Yaglom, *Geometric Transformations I, II, III*, The Mathematical Association of America, 1962.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr