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"A Study of Risāla al-Watar wa'l Jaib" ("The Treatise on the Chord and Sine"): Revisited

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Abstract. The purpose of this article is three-fold. First, we use Jamshīd al-Kāshī's cubic equation and *Mathematica* to calculate $\sin 1^\circ$ to over $11,200,000$ decimal digits of accuracy; an amazing improvement over the 17 digits of accuracy that al-Kāshī found by pencil and paper in 1426 [2]. Then, we conjecture that al-Kāshī's cubic equation can be used to calculate $\sin 1^\circ$ to any desired accuracy. Second, we set the record straight about the number of correct digits that al-Kāshī obtained. Third, we correct some statements that we made in [2].

1. Preliminaries

As we studied in detail [2], *Risāla al-watar wa'l jaib* ("The Treatise on the Chord and Sine"), is one of the three most significant mathematical achievements of the Iranian mathematician and astronomer Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī (1380-1429) that deals chiefly with the calculation of sine and chord of one-third of an angle with known sine and chord. As we discussed in [2], al-Kāshī applied Ptolemy's theorem to an inscribed quadrilateral to obtain his famous cubic equation, and then he used his cubic equation to calculate $\sin 1^\circ$ to 17 correct decimal digits as a root of his cubic equation.

The purpose of this article is three-fold. In Section 2, we use Jamshīd al-Kāshī's cubic equation that we discussed in [2] and *Mathematica* to calculate $\sin 1^\circ$ to over $11,200,000$ decimal digits of accuracy; an astonishing advancement over the 17 digits of accuracy that al-Kāshī found by pencil and paper in 1426 [2]. Also, in Section 2, we analyze our sexagesimal result in *Mathematica*. Moreover, we conjecture that al-Kāshī's cubic equation can be used to calculate $\sin 1^\circ$ to any desired accuracy. In Section 3, we set the record straight about the number of correct digits that al-Kāshī achieved. Finally, in Section 4, we correct some statements that we made in [2].

2. Al-Kāshī's cubic equation and *Mathematica*

We recall [2] that al-Kāshī used Ptolemy's theorem to obtain the following two forms of his famous cubic equation

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$$(i)\ \ x = \frac{a + x^3}{b}$$
$$(ii)\ \ x = \frac{4}{3}x^3 + \frac{1}{3}\sin 3^\circ.$$

In Section 4.2 of [2], we used (ii), his algorithm, and performed the calculation in decimal system to obtain the same level of accuracy for approximation of $\sin 1^\circ$ that was used for the value of $\sin 3^\circ$, namely, 17 decimal digits. In the following two lines of *Mathematica* program, we use the value of $\sin 3^\circ$ form the built-in function in *Mathematica* and al-Kāshī's cubic equation (ii) to obtain $11,200,000$ decimal digits of accuracy for $\sin 1^\circ$.

```
y = x /. NSolve[x - (4/3)*x^3 == (1/3) Sin[Pi/60], x, Reals, 11200000][[2]];
y - N[Sin[Pi/180], 11200000]
```

We note that the limitation of the accuracy of our calculation here is due to the weakness of the author's desktop computer's computational capability, and not al-Kāshī's cubic equation. Therefore, we can state the following conjecture.

Conjecture 2.1. Jamshīd al-Kāshī's cubic equation (ii) can be used to calculate $\sin 1^\circ$ to any desired accuracy in the decimal system, provided we use at least the same degree of accuracy for the value of $\sin 3^\circ$.

Remark 2.2. In Section 4.1 of [2], we used al-Kāshī's cubic equation (i), his algorithm, and $\sin 3^\circ$ with 9 digits of accuracy in sexagesimal system as he himself did to find an approximation for $\sin 1^\circ$ with 9 sexagesimal digits of accuracy as well. In the following four lines of *Mathematica* program, we use al-Kāshī's cubic equation (i) and the value of $\sin 3^\circ$ from the built-in function in *Mathematica* to obtain $10,912$ sexagesimal digits of accuracy for $\sin 1^\circ$.

```
$RecursionLimit = Infinity;
x[1] = 2;
x[n_] := N[x[n - 1]^3/(3*60^2) + 40*Sin[Pi/60], 19403]
RealDigits[x[19403]/120, 60, 10912] -RealDigits[Sin[Pi/180], 60, 10912]
```

Again, the limitation of the accuracy of our calculation here is due to the weakness of the author's desktop computer's computational capability in sexagesimal system. It is interesting to note that, as in [2] for every additional correct sexagesimal digit of $\sin 3^\circ$, we obtain an extra correct sexagesimal digit for $\sin 1^\circ$.

3. Let's set the record straight

In [2] we used al-Kāshī's cubic equation and his algorithm to find the value of $\sin 1^\circ$ in sexagesimal system as

$$1; 2, 49, 43, 11, 14, 44, 16, 26, 17.$$

This is equivalent to

$$\sin 1^\circ = 0.017452406437283510371 2,$$

in the decimal system, where the first 17 digits are correct. However, the last sexagesimal digit is not the same as the actual value of $\sin 1^\circ$. The actual value is 18 and not 17. Therefore, using al-Kāshī's cubic equation, the value of $\sin 3^\circ$ with 9 sexagesimal digits of accuracy, and his algorithm, we obtain only 9 sexagesimal digits of accuracy for $\sin 1^\circ$ and not 10.

As we stated in [2], the Arabic and the Persian manuscripts of *Risāla al-watar wa'l jaib* have been translated and/or commented on by various historians of mathematics and astronomy into English, French, German, and Russian. The number of correct sexagesimal digits of $\sin 1^\circ$ in these papers varies from 8 to 10 sexagesimal digits. Likewise, the number of correct decimal digits of $\sin 1^\circ$ in these papers ranges from 16 to 22 decimal digits. For example, Rosenfeld and Youschkevitch [3], stated that al-Kāshī obtained 10 sexagesimal digits and 18 decimal digits of accuracy. Aaboe [1], however, reported that al-Kāshī obtained only 8 sexagesimal digits of accuracy. Moreover, in the *Encyclopedia of Islam* [4], 16 decimal digits of accuracy is reported.

4. Corrections/comments regarding [2]

In this section, we correct some of the statements that we made in [2] as follows.

4.1. Footnote 7, p. 231. The author was deceived by claims in the literature that 'Abd al-'Alī Bīrjandī (d. 1528) was a student and colleague of both Jamshīd al-Kāshī and his cousin Mu'īn al-Dīn al-Kāshī. While Bīrjandī certainly looked at both al-Kāshī and his cousin as role models and they were a source of deep inspiration for him, he did not benefit from direct instruction from either one of them. One simple argument would be their age differences. Al-Kāshī died in 1429 while Bīrjandī died in 1528. Also, the village that Bīrjandī was born is actually *Bujd* and not *Wujd*.

4.2. Footnote 19, p. 238. The last line must be divided by 60. That is, the last line should be

$$1 \cdot 60^{-1} + 2 \cdot 60^{-2} + 49 \cdot 60^{-3} + 43 \cdot 60^{-4} + \cdots + 26 \cdot 60^{-9} + 17 \cdot 60^{-10}.$$

4.3. As we stated in Section 3, al-Kāshī's cubic equation (i) and his algorithm gave him only 9 sexagesimal digits of accuracy, which is equivalent to 17 decimal digits. In [2, line 17], the author erred in stating that al-Kāshī obtained 10 sexagesimal digits of accuracy.

4.4. Line 3, p. 240. (6) should be replaced by (13).

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