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A Construction of the Golden Ratio in an Arbitrary Triangle

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Abstract. We have known quite a lot about the construction of the golden ratio in the special triangles. In this article, we shall establish a construction of the golden ratio in arbitrary triangle using two symmedians and give a synthetic proof for this.

Given triangle ABC inscribed in circle (ω) center O.
(i) the symmedians AD and CF.
(ii) the ray DF meets (ω) at the point P.
(iii) the perpendicular line from P to OA meets AB and AC at Q and R respectively (See Figure 1).

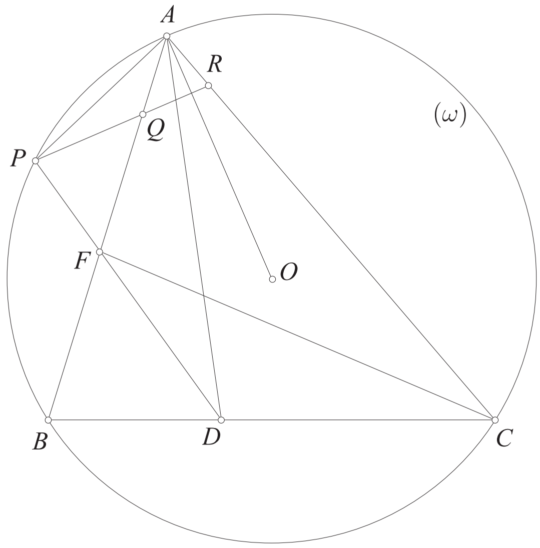

Figure 1

Proposition 1. *Q divides PR in the golden ratio.*

We give three lemmas to prove this proposition.

Lemma 2. *Given convex cyclic quadrilateral $ABCD$. Two diagonals AC and BD intersect at P. Then*

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$$\frac{PA}{PC} = \frac{AB \cdot AD}{CB \cdot CD}.$$

Proof. By inscribed angles are equal in the cyclic quadrilateral, we have the similar triangles $\triangle PAB \sim \triangle PDC$ and $\triangle PAD \sim \triangle PBC$. From this, we get the ratios

$$\frac{PA}{PB} = \frac{AD}{BC} \tag{1}$$

and

$$\frac{PB}{PC} = \frac{AB}{CD}. \tag{2}$$

From (1) and (2), we obtain

$$\frac{PA}{PC} = \frac{AB \cdot AD}{CB \cdot CD}.$$

This finishes the proof. $\square$

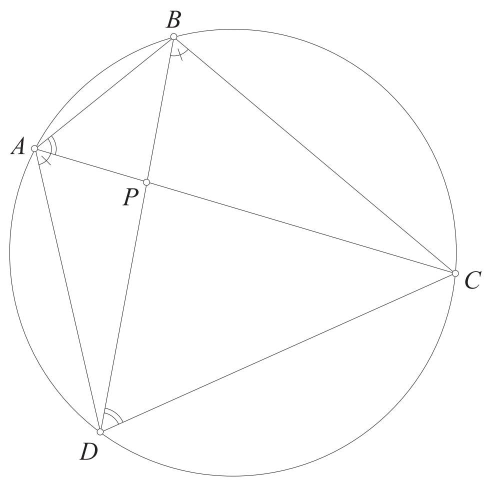

Figure 2

Lemma 3 (Ptolemy's Theorem [1])**.** *For a cyclic quadrilateral, the sum of the products of the two pairs of opposite sides equals the product of the diagonals.*

Using concept of homogeneous barycentric coordinates [9], we give and prove the following lemma

Lemma 4. *Let ABC be a triangle inscribed in circle (ω). P is a point inside triangle ABC. P has homogeneous barycentric coordinates $(x : y : z)$. DEF is cevian triangle of P. Ray EF meets (ω) at Q. Then*

$$\frac{CA}{yQB} = \frac{BC}{xQA} + \frac{AB}{zQC}.$$

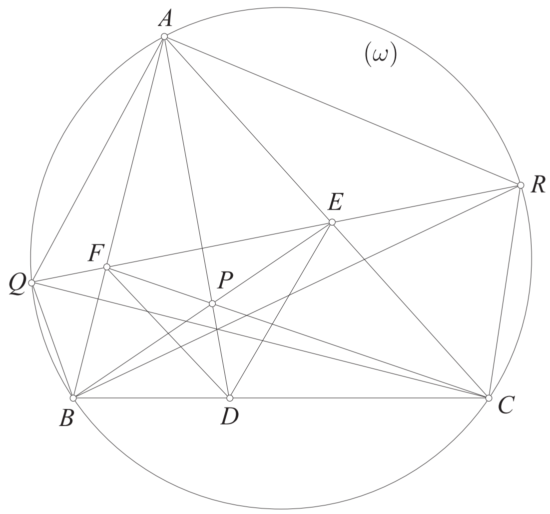

Figure 3

Proof. Because P has homogeneous barycentric coordinates $(x : y : z)$ so $E(x : 0 : z)$ and $F(x : y : 0)$. Thus we have the ratio

$$\frac{EA}{EC} = \frac{z}{x}. \tag{3}$$

Extend ray QE to meet (ω) again at R. From Lemma 2, we have

$$\frac{EA}{EC} = \frac{AQ \cdot AR}{CQ \cdot CR}. \tag{4}$$

From (3) and (4), we deduce

$$\frac{AQ \cdot AR}{CQ \cdot CR} = \frac{z}{x}.$$

Thus,

$$zQC = x\frac{AQ \cdot AR}{CR}. \tag{5}$$

Similarly, we have the identity

$$\frac{AQ \cdot AR}{BQ \cdot BR} = \frac{y}{x}.$$

Thus,

$$yQB = x\frac{AQ \cdot AR}{BR}. \tag{6}$$

Using (5) and (6) and Lemma 3, we consider the expression

$$\begin{aligned}\frac{CA}{yQB}-\frac{AB}{zQC}&=\frac{CA}{x\frac{AQ\cdot AR}{BR}}-\frac{AB}{x\frac{AQ\cdot AR}{CR}}\\&=\frac{CA\cdot RB-AB\cdot RC}{xAQ\cdot AR}\\&=\frac{BC\cdot RA}{xAQ\cdot AR}\\&=\frac{BC}{xQA}.\end{aligned}$$

Therefore

$$\frac{CA}{yQB}=\frac{BC}{xQA}+\frac{AB}{zQC}.$$

This completes the proof of Lemma 4 (See Figure 4). □

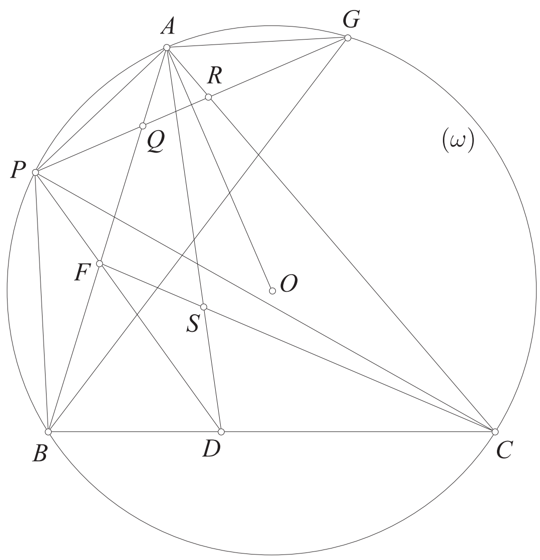

Figure 4

Proof of Proposition 1. The line PR meets (ω) again at G. From the similar triangles $\triangle QAP\sim\triangle QGB$ and $\triangle RAG\sim\triangle RPC$. We have the ratios

$$\frac{QA}{QP}=\frac{GA}{PB},\ \frac{RP}{RA}=\frac{PC}{GA}.$$

Therefore,

$$\frac{PR}{PQ}\cdot\frac{QA}{RA}=\frac{PC}{PB}. \tag{7}$$

Because QR is perpendicular to OA so QR is antiparallel line, this follows from $\angle AQR = \angle ACB$, thus $\triangle AQR \sim \triangle ACB$. We get

$$\frac{QA}{RA} = \frac{AC}{AB}. \tag{8}$$

From (7) and (8), we deduce

$$\frac{PR}{PQ} = \frac{AB \cdot PC}{CA \cdot PB}. \tag{9}$$

Let symmedians AD and CF meet at S then $S(BC^2 : CA^2 : AB^2)$ see [9]. Apply Lemma 4 for triangle ABC with S and ray DF meet (ω) at P, we have

$$\frac{BC}{BC^2 \cdot PA} = \frac{AB}{AB^2 \cdot PC} + \frac{CA}{CA^2 \cdot PB}.$$

This is equivalent to

$$\frac{1}{BC \cdot PA} = \frac{1}{AB \cdot PC} + \frac{1}{CA \cdot PB}. \tag{10}$$

Note that, by Lemma 3,

$$BC \cdot PA = AB \cdot PC - CA \cdot PB. \tag{11}$$

From (10) and (11) we have

$$\frac{1}{AB \cdot PC - CA \cdot PB} = \frac{1}{AB \cdot PC} + \frac{1}{CA \cdot PB}$$

This means

$$1 - \frac{CA \cdot PB}{AB \cdot PC} - \left(\frac{AB \cdot PC}{CA \cdot PB} - 1\right) = 1. \tag{12}$$

From (9) and (12), we obtain

$$\frac{PR}{PQ} - \frac{PQ}{PR} = 1.$$

This enough to show that the ratio

$$\frac{PR}{PQ} = \frac{\sqrt{5}+1}{2}$$

which is such the golden ratio. This completes the proof of Proposition 1.

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