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New Constructions of Triangle from $\alpha, b - c, t_A$

Martina Štěpánová

Abstract. We give two constructions of a triangle given its internal angle α, the length t_A of the angle bisector of α, and the difference $b - c$ of the side lengths b and c.

1. Introduction

In 2016 Paris Pamfilos published in this journal his solution of the following construction problem [3]. Let ABC be a triangle and let $a = |BC|$, $b = |CA|$, $c = |AB|$ denote the lengths of its sides. Let α be the interior angle at A, and $t_A = |AH|$, where H is the intersection of the side BC and the angle bisector of α. The problem is to construct the triangle ABC given α, $b - c$, t_A.

The key-point of Pamfilos' solution is the detection of a parabola which goes hand in hand with ABC and which is unambiguously constructible from the given data. In this paper, we present two other solutions. We think that our compass-and-straightedge constructions are simpler and shorter. We give two different verifications of the first construction. In the first verification of the first construction and in the second construction, any parabola does not play any role. In the second verification of the first construction, a parabola is considered, but not constructed at all.

In ABC, two cases can occur: either $b - c = 0$ or $b - c \neq 0$. If $b - c = 0$, then ABC is isosceles and its construction is trivial, since the side BC is perpendicular to the line-segment AH. If $b - c \neq 0$, then we assume without loss of generality that $b > c$.

2. First construction

First steps of the first construction are the same as in [3]. We recall not only them but also their justifications.

First steps of the first construction

Let M be the midpoint of the side BC. Consider a line parallel to AH that passes through M and denote its intersections with AB and AC by D and E, respectively (see Figure 1). Next, consider a line parallel to AH that passes through C and denote its intersection with AB by G.

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It is easily seen that $\angle AED = \angle ADE = \angle ACG = \angle AGC = \frac{\alpha}{2}$. Hence, triangles AED and ACG are isosceles and

$$|AD| = |AE| = |BD| - |AB| = \frac{|BG|}{2} - |AB| = \frac{b+c}{2} - c = \frac{b-c}{2}. \tag{1}$$

So, the first steps of the first construction are the following:

1 Draw a line-segment AH.
2 Draw lines m, n such that they form with AH angles $\frac{\alpha}{2}$.
3 Draw the isosceles triangle AED such that D lies on m, E lies on n, and $|AD| = |AE| = \frac{b-c}{2}$ (see Figure 1).

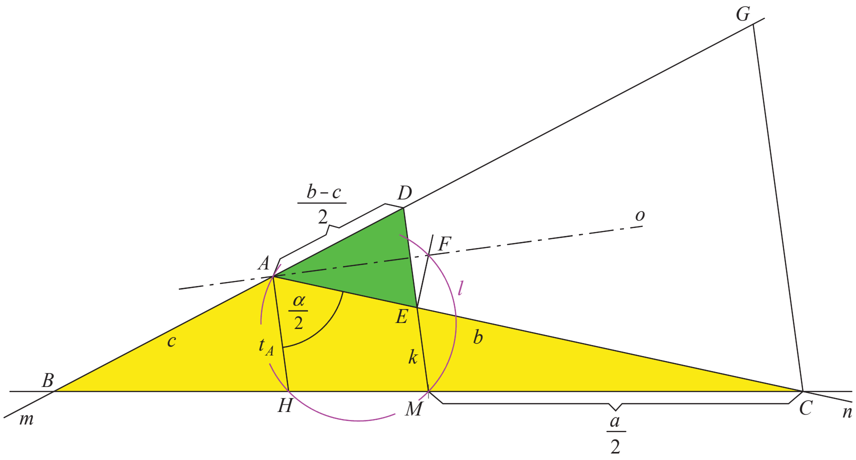

Figure 1.

2.1. *First approach.* Let F be the intersection of the perpendicular bisector of the line-segment DE with the line which is perpendicular to the line AE and passes through E. It is obvious that $\angle AFE = \angle AED = \frac{\alpha}{2}$, and therefore

$$|ED| = (b-c) \cdot \cos\frac{\alpha}{2}, \qquad |EF| = \frac{b-c}{2 \cdot \tan\frac{\alpha}{2}}\,. \tag{2}$$

Denote by k the length of the line-segment ME. Since the triangles AHC and EMC are similar, we obtain

$$\frac{t_A}{b} = \frac{k}{\frac{b+c}{2}}, \qquad \text{and, after rearranging,} \qquad t_A = \frac{2bk}{b+c}\,. \tag{3}$$

From the similarity of triangles BMD, BHA and from (2), we have

$$\frac{k + (b-c) \cdot \cos\frac{\alpha}{2}}{\frac{b+c}{2}} = \frac{t_A}{c},$$

and from (3), we obtain

$$k = \frac{t_A}{c} \cdot \frac{b+c}{2} - (b-c) \cdot \cos\frac{\alpha}{2} = \frac{bk}{c} - (b-c) \cdot \cos\frac{\alpha}{2}.$$

Therefore,

$$k\left(\frac{b}{c}-1\right)=(b-c)\cdot\cos\frac{\alpha}{2},$$

and thus

$$k=c\cdot\cos\frac{\alpha}{2}. \tag{4}$$

Set up a Cartesian coordinate system with A at the origin and the positive direction of the x-axis which coincides with the ray AC. Then (using (1), (2) and (4))

$$A=[0;\,0]\,,\qquad C=[b;\,0]\,,\qquad E=\left[\frac{b-c}{2};\,0\right],$$

$$F=\left[\frac{b-c}{2};\,\frac{b-c}{2\cdot\tan\frac{\alpha}{2}}\right],\qquad H=\left[t_A\cdot\cos\frac{\alpha}{2};\,-t_A\cdot\sin\frac{\alpha}{2}\right],$$

$$M=\left[\frac{b-c}{2}+k\cdot\cos\frac{\alpha}{2};\,-k\cdot\sin\frac{\alpha}{2}\right]=\left[\frac{b-c}{2}+c\cdot\cos^2\frac{\alpha}{2};\,-c\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}\right].$$

Next,

$$\overrightarrow{FM}=\left(c\cdot\cos^2\frac{\alpha}{2};\ -c\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}-\frac{b-c}{2\cdot\tan\frac{\alpha}{2}}\right),$$

$$\overrightarrow{CM}=\left(\frac{b-c}{2}-b+c\cdot\cos^2\frac{\alpha}{2};\,-c\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}\right).$$

It follows that

$$\begin{aligned}\overrightarrow{FM}\cdot\overrightarrow{CM}&=c\cdot\cos^2\frac{\alpha}{2}\left(\frac{b-c}{2}-b+c\cdot\cos^2\frac{\alpha}{2}\right)+\\&\quad+\left(c\cdot\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\frac{b-c}{2\cdot\tan\frac{\alpha}{2}}\right)\cdot c\cdot\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=\\&=c\cdot\cos^2\frac{\alpha}{2}\left(\frac{b-c}{2}-b+c\cdot\cos^2\frac{\alpha}{2}+c\cdot\sin^2\frac{\alpha}{2}+\frac{b-c}{2}\right)=\\&=c\cdot\cos^2\frac{\alpha}{2}\,(b-c-b+c)=0.\end{aligned}$$

Hence, the lines $CM=HM$ and FM are perpendicular, and, in consequence, the point M is the intersection of the line ED and the circle with HF as a diameter.

2.2. *Second approach.* Consider a parabola $\mathcal{P}$ to which the lines AE, AD, ED, BC are tangent. This parabola $\mathcal{P}$ is determined unambiguously ([1], p. 212). According to Lambert's theorem, the circumcircle of a triangle formed by three tangent lines to the parabola passes through the focus of the parabola. Thus, the circumcircles of the triangles AED, ABC and EMC pass through a single point (the focus of the parabola), which will be denoted by F (see Figure 2). It follows that $\angle BFC=\angle BAC=\alpha$ and $\angle MFC=\angle MEC=\frac{\alpha}{2}$. Hence, the focus F lies on the perpendicular bisector of the line-segment BC and the angles FMC, FMB are right. Since $\angle FEC=\angle FMC$, the angles FEC and FEA are also right. It implies that the line-segment AF is a diameter of the circumcircle of the isosceles triangle AED (and the point F is the same point as the point F in the first approach).

Since the angle FMH is right, the point M is the intersection of the line ED and the circle with diameter HF.

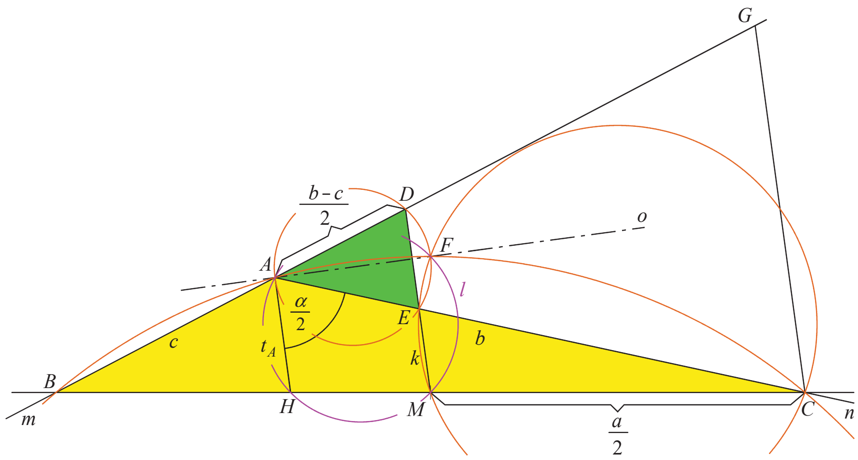

Figure 2.

Based on our previous considerations, we can now proceed with the remaining steps of our construction:

4a Draw the intersection F of the perpendicular bisector of DE with the line which is perpendicular to AE and passes through E, or,

4b draw the intersection F of the perpendicular bisector of DE with the circumcircle of the triangle AED.

5 Draw the intersection M of the line ED and the circle l with FH as a diameter (only the intersection M that lies in the half-plane determined by the line n and the point H can be taken into account).

6 Draw the intersection B of lines m and HM, and the intersection C of lines n and HM.

Note that the perpendicular bisector of DE is simultaneously the line passing through the point A and perpendicular to the line AH (and, of course, also the bisector of the angle EAD).

Comments on the solution published in [3]

We proceed with several comments on the solution presented in [3]. Paris Pamfilos considered the intersection of the perpendicular bisectors of the line-segments ED and BC. Clearly, this intersection is our point F (see Figure 3). Using Lambert's theorem he proved that the parabola $\mathcal{P}'$ with focus at F and whose tangent at its vertex is ED is the above-mentioned parabola $\mathcal{P}$, i.e. $\mathcal{P}'$=$\mathcal{P}$. He constructed the point F as the intersection of the perpendicular bisector of ED and the circumcircle of the triangle AED. Thus, the beginning of his solution coincides with our steps 1, 2, 3, 4b. But then, he constructed the line BC as the tangent to the parabola $\mathcal{P}'$ passing through H in a too complicated and time-consuming manner.

It is well known that the foot of the perpendicular line to any tangent of a parabola passing through its focus lies on the tangent at the vertex ([2], p. 34). Hence, the point M must lie on the circle with FH as a diameter. So, the construction can be completed using our very easy steps 5 and 6. Again, it is not necessary to construct any parabola.

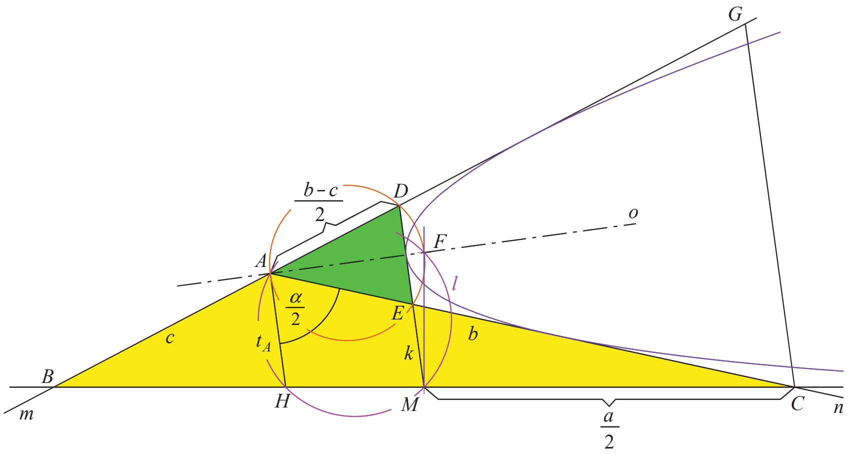

Figure 3.

3. Second construction

Let D' be a point on the ray opposite to the ray AB such that $|AD| = b - c$, i.e. $|BD'| = b$ (see Figure 4). Through this point D' draw a line parallel to AH and denote its intersections with lines AB, AC and BC by D', E' and M', respectively. Next, let

$$k' = \left|M'E'\right|, \qquad q = \frac{|D'E'|}{2}.$$

Since triangles $BM'D'$ and BHA are similar, we have

$$\frac{k' + 2q}{b} = \frac{t_A}{c}. \tag{5}$$

Analogously, from the similarity of triangles HCA and $M'CE'$, we obtain

$$\frac{t_A}{b} = \frac{k'}{c}, \qquad \text{and equivalently,} \qquad \frac{1}{b} = \frac{k'}{t_A \cdot c}. \tag{6}$$

Therefore (from (6) and (5)),

$$\frac{k'\left(k' + 2q\right)}{t_A \cdot c} = \frac{t_A}{c},$$

$$k'^2 + 2qk' - t_A^2 = 0.$$

This quadratic equation in the one unknown k' has two roots. But only the root $k' = -q + \sqrt{t_A^2 + q^2}$ is positive. If we consider a right triangle with legs of the lengths t_A and q, then $k' = -q + \sqrt{t_A^2 + q^2}$ is equal to the difference of the length $\sqrt{t_A^2 + q^2}$ of its hypotenuse and the length q of its leg.

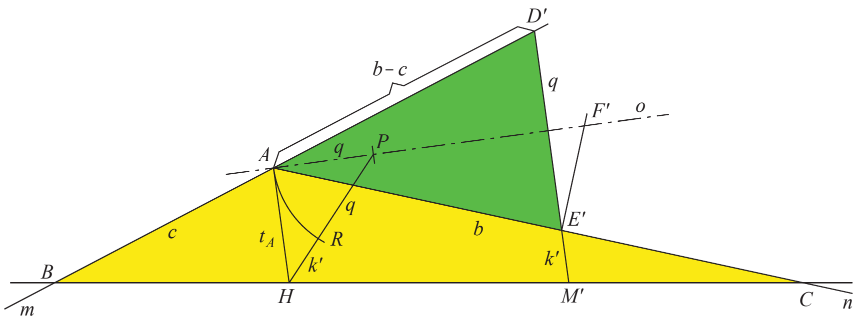

Figure 4.

Therefore, the steps of our second construction are the following (see Figure 4):

1 Draw a line-segment AH.
2 Draw lines m, n such that they form with AH angles $\frac{\alpha}{2}$.
3 Draw the isosceles triangle $AE'D'$ such that D' lies on m, E' lies on n, and $|AD'| = |AE'| = b - c$ (see Figure 4).
4 Draw the point P on the perpendicular bisector o of $D'E'$ such that $|AP| = \frac{|E'D'|}{2}$.
5 Draw the point R on the line-segment HP such that $|PR| = \frac{|E'D'|}{2}$.
6 Draw the point M' on the ray opposite to the ray $E'D'$ such that $|E'M'| = |RH|$.
7 Draw the intersection B of lines m and HM', and the intersection C of lines n and HM'.

Remark. If we consider the above-mentioned parabola $\mathcal{P}$, then the point E' is a point of $\mathcal{P}$ (therefore, AC touches $\mathcal{P}$ at E'). This follows from the fact that the sum of the lengths of subtangent and subnormal at any point of a parabola (except its vertex) is bisected by the focus of $\mathcal{P}$. Indeed, by comparing Figures 1 and 4, we observe that $|AF'| = 2\,|AF|$, where F' is the intersection of o and the line which is perpendicular to AC and passes through E'.

References

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Martina Štěpánová: Department of Mathematics Education, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic

E-mail address: `stepanov@karlin.mff.cuni.cz`