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# A New Proof of Pitot Theorem by AM-GM Inequality

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**Abstract**. In this note we show a new proof of Pitot theorem by Arithmetic Mean - Geometric Mean inequality. Our novel idea is to consider the before mentioned theorem as an extremal case of a general geometric inequality. This new approach cannot be found in the literature.

## 1. Introduction

The Pitot theorem states:

A quadrilateral $ABCD$ is tangential (has an inscribed circle) if and only if $AB + DC = AD + BC$.

To avoid reproduction we recommend to read the third and fourth paragraph of the excellent paper [1]. In this one the reader can find a brief survey on the theorem. Clearly we are interested in the converse of Pitot theorem, according to Martin Josefsson there are four proofs in the literature, providing references in each case. In this note we show still a new one, by means of the AM-GM inequality, since the well-known theorem can be considered an extremal case of a simple and general geometric inequality involving a quadrilateral, a circle, and tangents.

## 2. Lemma

Let $ABCD$ be a convex but not tangential quadrilateral. The extensions of $DA$ and $CB$ intersect at $E$, and the extensions of $BA$ and $CD$ intersect at $F$. Then the excircle of triangle $EAB$ cut the side $DC$ or the excircle of triangle $FAD$ cut the side $BC$.

*Proof.* In Figure 1, we draw the angle-bisectors of angles $E$ and $F$ respectively. Clearly points on the first line are equidistant from the sides $AD$ and $BC$, similar for the second one, where the points are equidistant from sides $AB$ and $DC$. Now, suppose the excircle of triangle $EAB$, is interior to the quadrilateral $ABCD$, hence its center $O_1$ is in the upper halfplane determined by the $F-$angle-bisector. But, $O_2$, the center of the second excircle is the intersection of the lines $AO_1$ and the $F-$angle-bisector, clearly this point is located in the right halfplane determined by the $E-$angle-bisector. Finally the excircle of triangle $FAD$ cut the side $BC$. □

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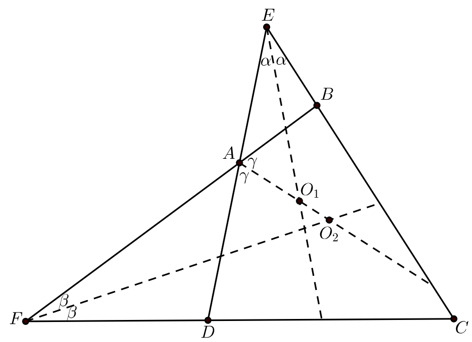


Figure 1

## 3. Converse of Pitot theorem

In this section we shall prove an stronger result than the converse of Pitot theorem. Let us see:

**Geometric Inequality:**

Let $ABCD$ be a quadrilateral with a circle that is tangent to the sides $AB, AD$ and $BC$ respectively, and cut the side $DC$. Then

$$AB + DC \geq AD + BC.$$

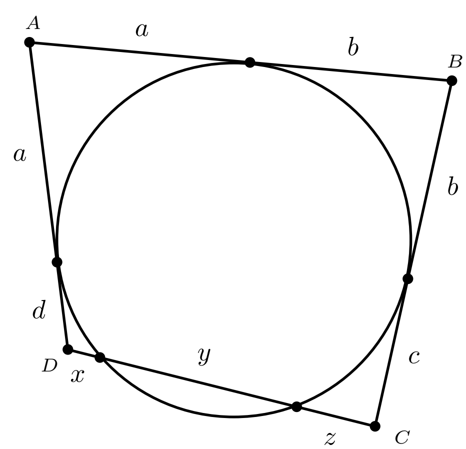


Figure 2

*Proof.* This circle exists by the Lemma.

See Figure 2, where the notation is completely justified by the *two tangent theorem*, that two tangents to a circle from an external point are of equal length. Now, the inequality to be proved becomes

$$x + y + z \geq c + d.$$

But, by Power of a Point theorem we have

$$\begin{aligned} c^2 &= z(y+z), \\ d^2 &= x(x+y). \end{aligned}$$

So, all that we need to prove is

$$\sqrt{z(y+z)} + \sqrt{x(x+y)} \leq x + y + z,$$

and this is a direct consequence of AM-GM inequality, since

$$\begin{aligned} \sqrt{z(y+z)} &\leq \frac{z+y+z}{2} = \frac{2z+y}{2}, \\ \sqrt{x(x+y)} &\leq \frac{x+x+y}{2} = \frac{2x+y}{2}, \end{aligned}$$

equality holds if and only if $y = 0$. Completing the proof of the inequality.

The geometric meaning of this result is that the circle is tangent to the side $DC$, i.e. the quadrilateral $ABCD$ is tangential. More precisely, to prove the converse of Pitot theorem, suppose the opposite sides add to the same number, and for sake of contradiction, the circle cut a side, then the inequality is strict, obtaining a contradiction. Done. □

**Bibliography.**

[1] M. Josefsson, More characterizations of tangential quadrilaterals, *Forum Geom.*, 11 (2011) 65–82.

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