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A Model of Continuous Plane Geometry that is Nowhere Geodesic

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Abstract. We construct a model $\mathbb{M}_1$ of plane geometry that satisfies all of Hilbert's axioms for the euclidean plane (with the exception of Sided-Angle-Side), yet in which the geodesic line segment connecting any two points A and B is never the shortest path from A to B. Moreover, the model $\mathbb{M}_1$ is continuous in the sense that it satisfies both the Ruler Postulate and Protractor Postulate from Birkhoff's set of axioms for the euclidean plane.

1. Introduction

In his talk given at the international congress of mathematicians in Paris in 1900, David Hilbert presented a list of problems. The fourth problem in this list is the following:

Problem of the straight line as the shortest distance between two points:

Another problem relating to the foundations of geometry is this : If from among the axioms necessary to establish ordinary euclidean geometry, we exclude the axiom of parallels, or assume it as not satisfied, but retain all other axioms, we obtain, as is well known, the geometry of Lobachevsky (hyperbolic geometry). We may therefore say that this is a geometry standing next to euclidean geometry. If we require further that that axiom be not satisfied whereby, of three points on a straight line, one and only one lies between the other two, we obtain Riemann's (elliptic) geometry, so that this geometry appears to be the next after Lobachevsky's. If we wish to carry out a similar investigation with respect to the axiom of Archimedes, we must look upon this as not satisfied, and we arrive thereby at the non-Archimedean geometries which have been investigated by Veronese and myself. The more general question now arises : Whether from other suggestive standpoints geometries may not be devised which, with equal right, stand next to euclidean geometry. Here I should like to direct your attention to a theorem which has, indeed, been employed by many authors as a definition of a straight line, viz., that the straight line is the shortest distance between two points. The essential content of this statement reduces to the theorem of Euclid that in a triangle the sum of two sides is always greater than the third side-a theorem which, as is easily seen, deals solely with elementary concepts, i. e., with such as are derived directly from the axioms, and

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is therefore more accessible to logical investigation. Euclid proved this theorem, with the help of the theorem of the exterior angle, on the basis of the congruence theorems. Now it is readily shown that this theorem of Euclid cannot be proved solely on the basis of those congruence theorems which relate to the application of segments and angles, but that one of the theorems on the congruence of triangles is necessary. We are asking, then, for a geometry in which all the axioms of ordinary euclidean geometry hold, and in particular all the congruence axioms except the one of the congruence of triangles (or all except the theorem of the equality of the base angles in the isosceles triangle), and in which, besides, the proposition that in every triangle the sum of two sides is greater than the third is assumed as a particular axiom.

One finds that such a geometry really exists and is no other than that which Minkowski constructed in his book, Geometrie der Zahlen [1] *and made the basis of his arithmetical investigations. Minkowski's is therefore also a geometry standing next to the ordinary euclidean geometry; it is essentially characterized by the following stipulations:*

1. The points which are at equal distances from a fixed point 0 lie on a convex closed surface of the ordinary euclidean space with 0 as a center.

2. Two segments are said to be equal when one can be carried into the other by a translation of the ordinary euclidean space.

In Minkowski's geometry the axiom of parallels also holds. By studying the theorem of the straight line as the shortest distance between two points, I arrived [2] *at a geometry in which the parallel axiom does not hold, while all other axioms of Minkowski' s geometry are satisfied. The theorem of the straight line as the shortest distance between two points and the essentially equivalent theorem of Euclid about the sides of a triangle, play an important part not only in number theory but also in the theory of surfaces and in the calculus of variations. For this reason, and because I believe that the thorough investigation of the conditions for the validity of this theorem will throw a new light upon the idea of distance, as well as upon other elementary ideas, e. g., upon the idea of the plane, and the possibility of its definition by means of the idea of the straight line, the construction and systematic treatment of the geometries here possible seem to me desirable.*

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There are various interpretations of Hilbert's fourth problem, some interpretations dealing with convex subsets of the Euclidean plane. The first main contribution to this problem was given by Hilbert's student G. Hamel. Hamel reduced the problem to metrics on convex subsets of Euclidean spaces [10]. In particular, if $\mathcal{S}$ is a convex subset of a Euclidean space, then the restriction to $\mathcal{S}$ of the ambient Euclidean metric is a metric on $\mathcal{S}$ satisfying the condition that the restriction to $\mathcal{S}$ of

[1]Leipzig, 1896.

[2]*Math Annalen*, Vol. 46, p.91.

the Euclidean straight lines are geodesics for the restriction to $\mathcal{S}$ of the Euclidean metric. One form of Hilbert's problem asks for a characterization of all metrics on $\mathcal{S}$ for which the Euclidean lines in $\mathcal{S}$ are geodesics [10].

Other interpretations deal with metrics in projective geometry [11]. Solutions to other interpretations are given by H. Busemann [2] and Z. I. Szabo [11]. There are many results related to this problem. For more information, see [2], [10], [11].

The contents of this paper were inspired by Hilbert's fourth problem. However, in this paper we look at the triangle inequality from the opposite point of view. In particular, we investigate plane geometry in which the straight line segment $\overline{AB}$ connecting two points A and B is never the shortest path from A to B. We give a model $\mathbb{M}_1$ of plane geometry in which all of the incidence axioms, betweenness axioms, congruence axioms (with the exception of Side-Angle-Side), and even the euclidean parallel postulate hold, yet in which the straight geodesic line segment $\overline{AB}$ connecting two points A and B is never the shortest path from A to B. We refer to such a model as *nowhere geodesic* since the geodesic line segment $\overline{AB}$ connecting two points A and B is never the shortest path from A to B. We prove that the following holds in $\mathbb{M}_1$:

Given any convex polygon $\mathcal{P}$, then for each pair of points A and B inside $\mathcal{P}$ and for any $\epsilon > 0$, there exists a path q from A to B such that the arc length along q is less than ϵ and such that q has nonempty intersection with the exterior of $\mathcal{P}$.

Thus, by going "outside" of $\mathcal{P}$, we can find shorter and shorter paths in $\mathbb{M}_1$ from A to B.

The model $\mathbb{M}_1$ is continuous in the sense that it satisfies both the ruler postulate and protractor postulate of Birkhoff [1], [7], [8], [9]. More specifically, all lines in $\mathbb{M}_1$ have a bijective correspondence with $\mathbb{R}$, and there is a bijective correspondence between all angles with fixed vertex V that are on a given halfplane of line $\overleftrightarrow{VP}$ and the open interval $(0, \pi)$.

We note that if all of the incidence axioms, betweenness axioms, and congruence axioms (with the possible exception of Side-Angle-Side) hold, and if in addition the exterior angle theorem, pons asinorum, and angle addition hold, then we can prove that the triangle inequality holds [4], [5], [7], [8]. When constructing the model $\mathbb{M}_1$, we use usual euclidean angle measure, so that both the exterior angle theorem and angle addition hold. However, due to the fact that distance is altered in $\mathbb{M}_1$, then the Pons Asinorun fails in the model.

In Taxicab geometry, both the exterior angle theorem and angle addition hold, but the pons asinorum fails. However, we still have that the general triangle inequality holds [3], [8]. Thus, in Taxicab Geometry, the straight line segment $\overline{AB}$ connecting two points A and B is still a shortest path from A to B.

2. Hilbert's Axioms

In this section we state the axioms of plane geometry given by Hilbert (as communicated by R. Hartshorne in [5]). We will show that the model $\mathbb{M}_1$ satisfies all of these axioms, with the exception of Side-Angle-Side.

The Incidence Axioms

(1) Given any two distinct points A and B, then there exists a unique line $\overleftrightarrow{AB}$ passing through A and B.
(2) Given any line l, then there exist at least two distinct points A and B on l.
(3) There exist three distinct noncollinear points A, B and C.

The Betweenness Axioms

(1) If B is between A and C (written $A - B - C$), then A, B, and C are three distinct collinear points. In this case we also have $C - B - A$.
(2) Given any two distinct points A and B, then there exists a point C such that $A - B - C$.
(3) Given three distinct points on a line, then exactly one of the three points is between the other two points.
(4) (Pasch) Let A, B, and C be three distinct noncollinear points, and let l be a line not passing through any of A, B, or C. If l passes through a point D lying between A and B, then either l passes through a point H lying between A and C, or else l passes through a point K lying between B and C, but not both.

We note that Betweenness Axiom (4) (i.e. Pasch) is logically equivalent to the Plane Separation Postulate stated below [4],[5],[7],[8]. Since Pasch and the Plane Separation Postulate are logically equivalent, then we will remove Pasch as an axiom and replace it with the Plane Separation Postulate.

The Plane Separation Postulate

Given any line l, then the set of points not lying on l can be divided into two nonempty subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ with the following properties:

(1) Two points A and B not on l belong to the same set ($\mathcal{H}_1$ or $\mathcal{H}_2$) if and only if segment $\overline{AB}$ does not intersect l.
(2) Two points C and D not on l belong to opposite sets ($C \in \mathcal{H}_1$ and $D \in \mathcal{H}_2$) if and only if segment $\overline{CD}$ intersects l at a point H such that $C - H - D$.

The sets $\mathcal{H}_1$ and $\mathcal{H}_2$ are called *halfplanes* (or *sides*) of the line l, and l is called an *edge* of each of the halfplanes $\mathcal{H}_1$ and $\mathcal{H}_2$. In case (1), we say that A and B are on the *same side* of l. In case (2), we say that C and D are on *opposite sides* of l. When quoting the Plane Separation Postulate, we will abbreviate it by PSP.

The Congruence Axioms for Line Segments

(1) Given a line segment $\overline{AB}$, and given a ray r originating at a point C, there exists a unique point D on the ray r such that $\overline{AB} \cong \overline{CD}$.
(2) If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Every line segment is congruent to itself.

(3) (Segment Addition) Given three points A, B, and C such that $A - B - C$, and given three points D, E, and F such that $D - E - F$, if $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.

The Congruence Axioms for Angles

(1) Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, then there exists a unique ray $\overrightarrow{DE}$ on a given side of line $\overleftrightarrow{DF}$ such that $\angle BAC \cong \angle EDF$.

(2) For any three angles α, β, and γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

The following is usually assumed as an axiom when working with Hilbert's axiom system for plane geometry. However, it does not hold in the model $\mathbb{M}_1$.

Side-Angle-Side

Given triangles $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

When referring to Side-Angle-Side, then we abbreviate it as *SAS*. It is well-known that if SAS holds, then one can prove the Exterior Angle Theorem, the Pons Asinorum, and Angle Addition as theorems, and consequently, one can prove that the Triangle Inequality also holds.

The following version of the Euclidean Parallel Postulate is by John Playfair (although it had previously been mentioned by Proclus) (page 39 of [5]):

Given a point P and a line l not passing through P, then there exists a unique line q such that q passes through P and is parallel to l.

3. The Ruler and Protractor Postulates

In this section we state two of the axioms given by Birkhoff in his development of plane geometry [1]. In particular, we state the ruler postulate and protractor postulate. Unlike the axioms of Hilbert given above, both the ruler postulate and protractor postulate incorporate the use of the real numbers through the concepts of distance and angle measure. Both of these postulates are satisfied by the model $\mathbb{M}_1$.

The Ruler Postulate

Let $\mathcal{P}$ denote the set of points in the plane. There exists a function $q : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ such that for each line l, there exists a bijection $f : l \to \mathbb{R}$ with the property that for all points P and Q on l, $q(P,Q) = |f(P) - f(Q)|$.

For all $P, Q \in \mathcal{P}$, we call $q(P,Q)$ the *distance* from P to Q. If for a line l, a bijection $f : l \to \mathbb{R}$ is such that for all $P, Q \in l$, $q(P,Q) = |f(P) - f(Q)|$, then f is called a *coordinate system* for l. We note that $q(P,Q)$ is not necessarily a metric on $\mathcal{P}$, and that $q(P,Q)$ does not necessarily satisfy the triangle inequality. In fact, we will define a distance function on the model $\mathbb{M}_1$ that is not a metric and that in general does not satisfy the triangle inequality.

The Protractor Postulate

There exists a function m from the set of all angles to the open interval $(0, \pi)$ such that

(1) For each ray $\overrightarrow{PQ}$ on the edge of halfplane $\mathcal{H}$ (where $\mathcal{H}$ is a halfplane of line $\overleftrightarrow{PQ}$), and for each $r \in (0, \pi)$, there exists a unique ray $\overrightarrow{PR}$, with $R \in \mathcal{H}$, such that $m(\angle QPR) = r$.

(2) If T is a point in the interior of $\angle QPR$, then $m(\angle QPT) + m(\angle TPR) = m(\angle QPR)$.

Given an angle $\angle ABC$, then $m(\angle ABC)$ is called the *measure* of angle $\angle ABC$, and is denoted by $m\angle ABC$.

4. The Model $\mathbb{M}_1$

To construct the model $\mathbb{M}_1$, we start with the Cartesian plane $\mathbb{R}^2$, and we define points and lines in $\mathbb{M}_1$ to be exactly the same as points and lines in $\mathbb{R}^2$. In particular, a point in $\mathbb{M}_1$ is given by an ordered pair of numbers (x, y), and a line is the set of all points in the plane satisfying an equation in one of the forms $y = mx + b$ or $x = k$. We also define angle measure in $\mathbb{M}_1$ to be exactly the same as euclidean angle measure in $\mathbb{R}^2$. Moreover, we define betweenness in $\mathbb{M}_1$ to be exactly the same as betweenness in euclidean geometry: point B is between points A and C in $\mathbb{M}_1$ (denoted $A - B - C$) if and only if B is between points A and C in $\mathbb{R}^2$. More specifically, if we let $d(X, Y)$ denote the euclidean distance from point X to point Y, then point B is between points A and C in both $\mathbb{M}_1$ and $\mathbb{R}^2$ if the following conditions hold:

(1) A, B, and C are distinct collinear points
(2) $d(A, C) = d(A, B) + d(B, C)$

Since points, lines, and betweenness in $\mathbb{M}_1$ are exactly the same as points, lines, and betweenness in euclidean geometry, then it follows immediately that the incidence and betweenness axioms hold in $\mathbb{M}_1$. Moreover, since points, lines, and betweenness in $\mathbb{M}_1$ are exactly the same as points, lines, and betweenness in euclidean geometry, then polygons in $\mathbb{M}_1$ are exactly the same as polygons in euclidean geometry. Also, since segments and polygons in $\mathbb{M}_1$ are the same as segments and polygons in euclidean geometry, then a polygon $\mathcal{P}$ is convex in $\mathbb{M}_1$ if and only if $\mathcal{P}$ is convex in euclidean geometry. Similarly, since angle measure in $\mathbb{M}_1$ is the same as euclidean angle measure, then it follows immediately that the congruence axioms for angles hold in $\mathbb{M}_1$. We note that since the Exterior Angle Theorem and Angle Addition are based solely on angle measure and not on distance, and since both of these statements hold in euclidean geometry, then they both hold in $\mathbb{M}_1$. However, to construct $\mathbb{M}_1$, we will need to alter distance in the plane, and as a consequence the Pons Asinorum does not hold in $\mathbb{M}_1$. Furthermore, since we are altering distance, then we will need to prove that the congruence axioms for line segments hold in $\mathbb{M}_1$. To define distance in $\mathbb{M}_1$, we start with the set of points (u, v) in the plane such that $u, v \in \mathbb{Q}$. That is, we start with the cartesian product $\mathbb{Q}^2$. Since $\mathbb{Q}^2$ is countably infinite, then we can enumerate the points $P_1, P_2, \ldots, P_j, \ldots$ in $\mathbb{Q}^2$. Thus, we often will treat the points in $\mathbb{Q}^2$ as a sequence (P_j). Using this enumeration on the set of all points (P_j) in $\mathbb{Q}^2$, we can enumerate all (non-ordered) pairs of distinct points from $\mathbb{Q}^2$ in the following way: Given the pairs of points $(P_l, P_t), (P_h, P_k) \in \mathbb{Q}^2 \times \mathbb{Q}^2$, where $P_l \neq P_t$ and $P_h \neq P_k$, then

assume that k is the largest of the four subscripts l, t, h, and k. If k is strictly larger than the other subscripts l, t, and h, then (P_l, P_t) comes before (P_h, P_k) in the enumeration. If $k = t$, then we compare l and h. In this case, (P_l, P_t) comes before (P_h, P_k) if and only if $l < h$. Similarly, if $k = l$, then (P_l, P_t) comes before (P_h, P_k) if and only if $t < h$. For example, the first ten pairs are enumerated in the following way:
(P_1, P_2), (P_1, P_3), (P_2, P_3), (P_1, P_4), (P_2, P_4), (P_3, P_4), (P_1, P_5), (P_2, P_5), (P_3, P_5), (P_4, P_5),

Note that the order in which the points in each pair are written is irrelevant when enumerating the pairs of points this way. In particular, (P_l, P_t) is considered the same as (P_t, P_l) for the purposes of enumeration.

Again, we let $d(X, Y)$ denote the euclidean distance from a point X to a point Y. We let $n(X, Y)$ denote the distance in the model $\mathbb{M}_1$ from a point X to a point Y. For each pair of distinct points P_r and P_t in $\mathbb{Q}^2$, with $r < t$, we construct a sequence $(D_{r,t}(m))$ of points in $\mathbb{R}^2$ such that for each $m \geq 1$, $d(P_r, D_{r,t}(m)) \geq m$ and $d(P_t, D_{r,t}(m)) \geq m$, and such that there exists a path from P_r to P_t passing through $D_{r,t}(m)$ whose arc length in $\mathbb{M}_1$ is equal to $\dfrac{n(P_r, P_t)}{2^m}$.

Let P_1 and P_2 be the first two points in the sequence (P_j). If X and Y are points on the line $\overleftrightarrow{P_1P_2}$, then we define $n(X, Y) = d(X, Y)$. In particular, $n(P_1, P_2) = d(P_1, P_2)$.

Let l_1 be a line perpendicular to $\overleftrightarrow{P_1P_2}$. Since l_1 is unbounded in euclidean geometry, then there exists a point $D_{1,2}(1)$ on l_1 such that

(1) $D_{1,2}(1) \notin \overleftrightarrow{P_1P_2}$
(2) $d(P_1, D_{1,2}(1)) \geq 1$ and $d(P_2, D_{1,2}(1)) \geq 1$

In particular, the three points $D_{1,2}(1)$, P_1, and P_2 are not collinear. We define $n(P_1, D_{1,2}(1)) = n(P_2, D_{1,2}(1)) = \dfrac{1}{4}n(P_1, P_2)$. We see that the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1D_{1,2}(1)}$ and $\overline{P_2D_{1,2}(1)}$ is $\dfrac{1}{2}n(P_1, P_2)$.

We now show how to compute the distance in $\mathbb{M}_1$ along the line $\overleftrightarrow{P_1D_{1,2}(1)}$ between a point $P \in \overleftrightarrow{P_1D_{1,2}(1)}$ and either of the points P_1 or $D_{1,2}(1)$. We have three cases.

First assume that $P - P_1 - D_{1,2}(1)$. In this case, $n(P, P_1) = d(P, P_1)$ and $n(P, D_{1,2}(1)) = n(P, P_1) + n(P_1, D_{1,2}(1)) = d(P, P_1) + \dfrac{1}{4}n(P_1, P_2)$.

Next, assume that $P_1 - P - D_{1,2}(1)$. In this case, there exist $r_1, r_2 \in (0, 1)$ such that $d(P_1, P) = r_1 d(P_1, D_{1,2}(1))$ and $d(P, D_{1,2}(1)) = r_2 d(P_1, D_{1,2}(1))$. We define $n(P_1, P) = r_1 n(P_1, D_{1,2}(1))$ and $n(P, D_{1,2}(1)) = r_2 n(P_1, D_{1,2}(1))$. Note that we use the same constants of proportionality r_1 and r_2 for both euclidean geometry and $\mathbb{M}_1$.

Finally, assume that $P_1 - D_{1,2}(1) - P$. In this case, $n(D_{1,2}(1), P) = d(D_{1,2}(1), P)$ and $n(P_1, P) = n(P_1, D_{1,2}(1)) + n(D_{1,2}(1), P) = \dfrac{1}{4}n(P_1, P_2) + d(D_{1,2}(1), P)$.

We can now use these definitions of length in $\mathbb{M}_1$ together with the use of segment addition and constants of proportionality to define the distance in $\mathbb{M}_1$ along the line $\overleftrightarrow{P_1D_{1,2}(1)}$ between any two points X_1 and X_2 on $\overleftrightarrow{P_1D_{1,2}(1)}$.

If $X_1 - X_2 - P_1 - D_{1,2}(1)$, then $n(X_1, X_2) = d(X_1, X_2)$. Similarly, if $P_1 - D_{1,2}(1) - X_1 - X_2$, then $n(X_1, X_2) = d(X_1, X_2)$.

Assume that $P_1 - X_1 - X_2 - D_{1,2}(1)$. In this case, there exists $r_3 \in (0, 1)$ such that $d(X_1, X_2) = r_3 d(P_1, D_{1,2}(1))$. We define $n(X_1, X_2) = r_3 n(P_1, D_{1,2}(1))$.

Next, assume that $X_1 - P_1 - D_{1,2}(1) - X_2$. In this case, we define $n(X_1, X_2)$ = $n(X_1, P_1) + n(P_1, D_{1,2}(1)) + n(D_{1,2}(1), X_2)$ = $d(X_1, P_1) + n(P_1, D_{1,2}(1)) + d(D_{1,2}(1), X_2)$.

Next, assume that $X_1 - P_1 - X_2 - D_{1,2}(1)$. In this case, we define $n(X_1, X_2)$ = $n(X_1, P_1) + n(P_1, X_2)$.

Finally, assume that $P_1 - X_1 - D_{1,2}(1) - X_2$. Similar to the previous case, we define $n(X_1, X_2)$ = $n(X_1, D_{1,2}(1)) + n(D_{1,2}(1), X_2)$.

One can use the same exact method to define the distance in $\mathbb{M}_1$ along the line $\overleftrightarrow{P_2D_{1,2}(1)}$ between any two points on $\overleftrightarrow{P_2D_{1,2}(1)}$.

Let $\mathcal{S}_{1,2}(1)$ denote the set of lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1D_{1,2}(1)}$, and $\overleftrightarrow{P_2D_{1,2}(1)}$. Let l_2 be a line not in $\mathcal{S}_{1,2}(1)$. Since $\mathcal{S}_{1,2}(1)$ is finite, and since l_2 is unbounded and has infinitely many points, then there exists a point $D_{1,2}(2)$ on l_2 such that

(1) $d(P_1, D_{1,2}(2)) \geq 2$ and $d(P_2, D_{1,2}(2)) \geq 2$
(2) $D_{1,2}(2)$ is not on any of the lines in $\mathcal{S}_{1,2}(1)$.

Note that $D_{1,2}(2) \neq D_{1,2}(1)$, and that no three of P_1, P_2, $D_{1,2}(1)$, or $D_{1,2}(2)$ are collinear. Thus, the distance in $\mathbb{M}_1$ between $D_{1,2}(2)$ and either of P_1 and P_2 has not yet been defined.

We define $n(P_1, D_{1,2}(2)) = n(P_2, D_{1,2}(2)) = \dfrac{1}{8} n(P_1, P_2) = \dfrac{1}{8} d(P_1, P_2)$. We see that the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1D_{1,2}(2)}$ and $\overline{P_2D_{1,2}(2)}$ is $\dfrac{1}{4} n(P_1, P_2)$.

Using a method similar to the one given above to define distance in $\mathbb{M}_1$ along $\overleftrightarrow{P_1D_{1,2}(1)}$ between any two points on $\overleftrightarrow{P_1D_{1,2}(1)}$, one can similarly define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1D_{1,2}(2)}$ or between any two points on the line $\overleftrightarrow{P_2D_{1,2}(2)}$.

Let $\mathcal{S}_{1,2}(2)$ denote the set of lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1D_{1,2}(1)}$, $\overleftrightarrow{P_2D_{1,2}(1)}$, $\overleftrightarrow{P_1D_{1,2}(2)}$, $\overleftrightarrow{P_2D_{1,2}(2)}$, and $\overleftrightarrow{D_{1,2}(1)D_{1,2}(2)}$. Let l_3 be a line not in $\mathcal{S}_{1,2}(2)$. Since $\mathcal{S}_{1,2}(2)$ is finite, and since l_3 is unbounded and has infinitely many points, then there exists a point $D_{1,2}(3)$ on l_3 such that

(1) $d(P_1, D_{1,2}(3)) \geq 3$ and $d(P_2, D_{1,2}(3)) \geq 3$
(2) $D_{1,2}(3)$ is not on any of the lines in $\mathcal{S}_{1,2}(2)$.

Note that $D_{1,2}(3)$ is distinct from $D_{1,2}(1)$ and $D_{1,2}(2)$, and that no three of P_1, P_2, $D_{1,2}(1)$, $D_{1,2}(2)$, or $D_{1,2}(3)$ are collinear. Thus, the distance in $\mathbb{M}_1$ between $D_{1,2}(3)$ and either of P_1 and P_2 has not yet been defined.

We define $n(P_1, D_{1,2}(3)) = n(P_2, D_{1,2}(3)) = \dfrac{1}{16}n(P_1, P_2) = \dfrac{1}{16}d(P_1, P_2)$. We see that the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1D_{1,2}(3)}$ and $\overline{P_2D_{1,2}(3)}$ is $\dfrac{1}{8}n(P_1, P_2)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ along $\overleftrightarrow{P_1D_{1,2}(1)}$ between any two points on $\overleftrightarrow{P_1D_{1,2}(1)}$, one can similarly define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1D_{1,2}(3)}$ or between any two points on the line $\overleftrightarrow{P_2D_{1,2}(3)}$.

Assume that there exist k distinct points $D_{1,2}(1), D_{1,2}(2), \dots, D_{1,2}(k) \in \mathbb{R}^2$ such that:

(1) No three of the points $P_1, P_2, D_{1,2}(1), D_{1,2}(2), \dots, D_{1,2}(k)$ are collinear.
(2) For each $t = 1, \dots k$, $d(P_1, D_{1,2}(t)) \geq t$ and $d(P_2, D_{1,2}(t)) \geq t$
(3) For each $t = 1, \dots k$, $n(P_1, D_{1,2}(t)) = n(P_2, D_{1,2}(t)) = \dfrac{1}{2^{t+1}}n(P_1, P_2)$
(4) For each $t = 1, \dots k$, we use a method similar to the one used above to define distance in $\mathbb{M}_1$ between two points on the line $\overleftrightarrow{P_1D_{1,2}(t)}$ or between any two points on the line $\overleftrightarrow{P_2D_{1,2}(t)}$

We note that for each $t = 1, \dots k$, the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1D_{1,2}(t)}$ and $\overline{P_2D_{1,2}(t)}$ is $\dfrac{1}{2^t}n(P_1, P_2)$.

Let $\mathcal{S}_{1,2}(k)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1P_2}$
(2) For each $t = 1, \dots k$, $\overleftrightarrow{P_1D_{1,2}(t)}$ and $\overleftrightarrow{P_2D_{1,2}(t)}$
(3) For each $t, h \in \{1, \dots k\}$, with $t < h$, $\overleftrightarrow{D_{1,2}(t)D_{1,2}(h)}$

Let l_{k+1} be a line not in $\mathcal{S}_{1,2}(k)$. Since $\mathcal{S}_{1,2}(k)$ is finite, and since l_{k+1} is unbounded and has infinitely many points, then there exists a point $D_{1,2}(k+1)$ on l_{k+1} such that

(1) $d(P_1, D_{1,2}(k+1)) \geq k+1$ and $d(P_2, D_{1,2}(k+1)) \geq k+1$
(2) $D_{1,2}(k+1)$ is not on any of the lines in $\mathcal{S}_{1,2}(k)$.

Note that $D_{1,2}(k+1)$ is distinct from all of the points $D_{1,2}(1), D_{1,2}(2), \dots, D_{1,2}(k)$, and that no three of $P_1, P_2, D_{1,2}(1), D_{1,2}(2), \dots, D_{1,2}(k), D_{1,2}(k+1)$ are collinear. Thus, the distance in $\mathbb{M}_1$ between $D_{1,2}(k+1)$ and either of P_1 and P_2 has not yet been defined. We define $n(P_1, D_{1,2}(k+1)) = n(P_2, D_{1,2}(k+1)) = \dfrac{1}{2^{k+2}}n(P_1, P_2)$.

Again, using a method similar to the one used above, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1D_{1,2}(k+1)}$ or between any two points on the line $\overleftrightarrow{P_2D_{1,2}(k+1)}$.

We note that the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1D_{1,2}(k+1)}$ and $\overline{P_2D_{1,2}(k+1)}$ is $\dfrac{1}{2^{k+1}}n(P_1, P_2)$.

Thus, there exists a sequence $(D_{1,2}(m))$ of distinct points in $\mathbb{R}^2$ such that

(1) No three points in the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 1\}$ are collinear

(2) For each $m \geq 1$, $d(P_1, D_{1,2}(m)) \geq m$ and $d(P_2, D_{1,2}(m)) \geq m$
(3) For each $m \geq 1$, $n(P_1, D_{1,2}(m)) = n(P_2, D_{1,2}(m)) = \dfrac{1}{2^{m+1}} n(P_1, P_2)$.
(4) For each $m \geq 1$, we use a method similar to the one used above to define distance in $\mathbb{M}_1$ between two points on the line $\overleftrightarrow{P_1 D_{1,2}(m)}$ or between any two points on the line $\overleftrightarrow{P_2 D_{1,2}(m)}$.

We note that for each $m \geq 1$, the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1 D_{1,2}(m)}$ and $\overline{P_2 D_{1,2}(m)}$ is $\dfrac{1}{2^m} n(P_1, P_2)$.

Let P_3 denote the third point in the sequence (P_j). If P_3 is on $\overleftrightarrow{P_1 P_2}$ or if there exists $m \geq 1$ such that P_3 is on $\overleftrightarrow{P_1 D_{1,2}(m)}$ then $n(P_1, P_3)$ has already been defined. More generally, if P_3 is on $\overleftrightarrow{P_1 P_2}$ or if there exists $m \geq 1$ such that P_3 is on $\overleftrightarrow{P_1 D_{1,2}(m)}$ then for each pair of points $X, Y \in \overleftrightarrow{P_1 P_3}$, the distance $n(X, Y)$ has already been defined. On the other hand, if P_3 is not on $\overleftrightarrow{P_1 P_2}$ and if for each $m \geq 1$, P_3 is not on $\overleftrightarrow{P_1 D_{1,2}(m)}$ then $n(P_1, P_3)$ has not yet been defined. In this case, we define $n(P_1, P_3) = d(P_1, P_3)$, and in general, given $X, Y \in \overleftrightarrow{P_1 P_3}$, we define $n(X, Y) = d(X, Y)$. In either case, $n(P_1, P_3)$ has been defined, and more generally, distance along $\overleftrightarrow{P_1 P_3}$ has been defined.

Let $\mathcal{S}_{1,3}(1)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_1 P_3}$, and $\overleftrightarrow{P_2 P_3}$
(2) For each $m \geq 1$, $\overleftrightarrow{P_1 D_{1,2}(m)}$ and $\overleftrightarrow{P_2 D_{1,2}(m)}$
(3) For each t, h, with $h \geq 3$ and $t < h$, $\overleftrightarrow{D_{1,2}(t) D_{1,2}(h)}$.

Since no three points in the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 1\}$ are collinear, then none of the points from the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 3\}$ are on $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$. In particular, $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)} \notin \mathcal{S}_{1,3}(1)$. Since there are only a countably infinite number of lines that are elements in $\mathcal{S}_{1,3}(1)$, then there are at most a countably infinite number of points of intersection of any of the lines in $\mathcal{S}_{1,3}(1)$ with the line $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$. Since $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$ is unbounded, and since there exist an uncountably infinite number of points between any two distinct points on $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$, then there exists a point $D_{1,3}(1)$ on $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$ such that:

(1) $d(P_1, D_{1,3}(1)) \geq 1$ and $d(P_3, D_{1,3}(1)) \geq 1$
(2) $D_{1,3}(1)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(1)$.

Since $D_{1,3}(1)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(1)$, then neither $n(P_1, D_{1,3}(1))$ nor $n(P_3, D_{1,3}(1))$ has yet been defined. We define $n(P_1, D_{1,3}(1)) = n(P_3, D_{1,3}(1)) = \dfrac{1}{4} n(P_1, P_3)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1 D_{1,3}(1)}$ or between any two points on the line $\overleftrightarrow{P_3 D_{1,3}(1)}$.

Also, for any $X, Y \in \overleftrightarrow{P_2D_{1,3}(1)}$ we define $n(X,Y) = d(X,Y)$. Moreover, for each $m \geq 3$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m)D_{1,3}(1)}$, we define $n(X,Y) = d(X,Y)$.

Let $\mathcal{S}_{1,3}(2)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, and $\overleftrightarrow{P_2P_3}$
(2) For each $m \geq 1$, $\overleftrightarrow{P_1D_{1,2}(m)}$, $\overleftrightarrow{P_2D_{1,2}(m)}$, and $\overleftrightarrow{D_{1,2}(m)D_{1,3}(1)}$
(3) $\overleftrightarrow{P_1D_{1,3}(1)}$, $\overleftrightarrow{P_2D_{1,3}(1)}$, and $\overleftrightarrow{P_3D_{1,3}(1)}$
(4) Any line in the form $\overleftrightarrow{D_{1,2}(t)D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$.

Since no three points in the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 1\}$ are collinear, then none of the points from the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \notin \{2,3\}\}$ are on $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$. Moreover, by the construction above, we have that the point $D_{1,3}(1)$ is not on line $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$. Thus, $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)} \notin \mathcal{S}_{1,3}(2)$.

Since there are only a countably infinite number of lines that are elements in $\mathcal{S}_{1,3}(2)$, then there are at most a countably infinite number of points of intersection of any of the lines in $\mathcal{S}_{1,3}(2)$ with the line $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$. Since $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$ is unbounded, and since there exist an uncountably infinite number of points between any two distinct points on $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$, then there exists a point $D_{1,3}(2)$ on $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$ such that:

(1) $d(P_1, D_{1,3}(2)) \geq 2$ and $d(P_3, D_{1,3}(2)) \geq 2$
(2) $D_{1,3}(2)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(2)$.

Since $D_{1,3}(2)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(2)$, then neither $n(P_1, D_{1,3}(2))$ nor $n(P_3, D_{1,3}(2))$ has yet been defined. We define $n(P_1, D_{1,3}(2)) = n(P_3, D_{1,3}(2)) = \frac{1}{8}n(P_1, P_3)$.

Thus, the arc length of the path from P_1 to P_3 along the segments $\overline{P_1D_{1,3}(2)}$ and $\overline{P_3D_{1,3}(2)}$ is $\frac{1}{4}n(P_1, P_3)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1D_{1,2}(1)}$, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1D_{1,3}(2)}$ or between any two points on the line $\overleftrightarrow{P_3D_{1,3}(2)}$.

Also, for any $X, Y \in \overleftrightarrow{P_2D_{1,3}(2)}$ we define $n(X,Y) = d(X,Y)$. Moreover, for each $m \neq 2, 3$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m)D_{1,3}(2)}$, we define $n(X,Y) = d(X,Y)$.

Let $\mathcal{S}_{1,3}(3)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, and $\overleftrightarrow{P_2P_3}$
(2) For each $m \geq 1$, $\overleftrightarrow{P_1D_{1,2}(m)}$, $\overleftrightarrow{P_2D_{1,2}(m)}$, $\overleftrightarrow{D_{1,2}(m)D_{1,3}(1)}$, and $\overleftrightarrow{D_{1,2}(m)D_{1,3}(2)}$
(3) $\overleftrightarrow{P_1D_{1,3}(1)}$, $\overleftrightarrow{P_2D_{1,3}(1)}$, $\overleftrightarrow{P_3D_{1,3}(1)}$, $\overleftrightarrow{P_1D_{1,3}(2)}$, $\overleftrightarrow{P_2D_{1,3}(2)}$, $\overleftrightarrow{P_3D_{1,3}(2)}$, and $\overleftrightarrow{D_{1,3}(1)D_{1,3}(2)}$

(4) Any line in the form $\overleftrightarrow{D_{1,2}(t)D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$.

Since no three points in the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 1\}$ are collinear, then none of the points from the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \notin \{3,4\}\}$ are on $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$. Moreover, by the constructions above, we have that neither of the points $D_{1,3}(1)$ or $D_{1,3}(2)$ are on line $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$. Thus, $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)} \notin \mathcal{S}_{1,3}(3)$.

Since there are only a countably infinite number of lines that are elements in $\mathcal{S}_{1,3}(3)$, then there are at most a countably infinite number of points of intersection of any of the lines in $\mathcal{S}_{1,3}(3)$ with the line $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$. Since $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$ is unbounded, and since there exist an uncountably infinite number of points between any two distinct points on $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$, then there exists a point $D_{1,3}(3)$ on $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$ such that:

(1) $d(P_1, D_{1,3}(3)) \geq 3$ and $d(P_3, D_{1,3}(3)) \geq 3$
(2) $D_{1,3}(3)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(3)$.

Since $D_{1,3}(3)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(3)$, then neither $n(P_1, D_{1,3}(3))$ nor $n(P_3, D_{1,3}(3))$ has yet been defined. We define $n(P_1, D_{1,3}(3)) = n(P_3, D_{1,3}(3)) = \dfrac{1}{16} n(P_1, P_3)$.

Thus, the arc length of the path from P_1 to P_3 along the segments $\overline{P_1 D_{1,3}(3)}$ and $\overline{P_3 D_{1,3}(3)}$ is $\dfrac{1}{8} n(P_1, P_3)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1 D_{1,3}(3)}$ or between any two points on the line $\overleftrightarrow{P_3 D_{1,3}(3)}$.

Also, for any $X, Y \in \overleftrightarrow{P_2 D_{1,3}(3)}$ we define $n(X,Y) = d(X,Y)$. Moreover, for each $m \neq 3, 4$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m) D_{1,3}(3)}$, we define $n(X,Y) = d(X,Y)$.

Assume that there exist r distinct points $D_{1,3}(1)$, $D_{1,3}(2)$, $D_{1,3}(3)$, …, $D_{1,3}(r)$ (where $r \geq 3$) such that

(1) For each $j = 1, \ldots r$, $D_{1,3}(j)$ is a point on the line $\overleftrightarrow{D_{1,2}(j)D_{1,2}(j+1)}$.
(2) No three of the points $D_{1,3}(1)$, $D_{1,3}(2)$, $D_{1,3}(3)$, …, $D_{1,3}(r)$ are collinear.
(3) For each $j = 1, \ldots r$, $D_{1,3}(j)$ is not a point on any of the following lines:
    (a) $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, and $\overleftrightarrow{P_2P_3}$
    (b) For each $m \geq 1$, $\overleftrightarrow{P_1 D_{1,2}(m)}$ and $\overleftrightarrow{P_2 D_{1,2}(m)}$
    (c) For each $k \neq j$, $\overleftrightarrow{P_1 D_{1,3}(k)}$, $\overleftrightarrow{P_2 D_{1,3}(k)}$, and $\overleftrightarrow{P_3 D_{1,3}(k)}$
    (d) For each $m \geq 1$, and for each $k \neq j$, $\overleftrightarrow{D_{1,2}(m) D_{1,3}(k)}$
    (e) Any line in the form $\overleftrightarrow{D_{1,2}(t) D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(j) D_{1,2}(j+1)}$
    (f) Any line in the form $\overleftrightarrow{D_{1,3}(t) D_{1,3}(h)}$, where $t \neq j$ and $h \neq j$.

(4) For each $j = 1, \dots r$, $d(P_1, D_{1,3}(j)) \geq j$ and $d(P_3, D_{1,3}(j)) \geq j$

(5) For each $j = 1, \dots r$, $n(P_1, D_{1,3}(j)) = n(P_3, D_{1,3}(j)) = \dfrac{1}{2^{j+1}} n(P_1, P_3)$.

(6) For each $j = 1, \dots r$, distance in $\mathbb{M}_1$ is defined between any two points on the line $\overleftrightarrow{P_1 D_{1,3}(j)}$ or between any two points on the line $\overleftrightarrow{P_3 D_{1,3}(j)}$ using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$.

Let $\mathcal{S}_{1,3}(r+1)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_1 P_3}$, and $\overleftrightarrow{P_2 P_3}$

(2) For each $m \geq 1$, $\overleftrightarrow{P_1 D_{1,2}(m)}$ and $\overleftrightarrow{P_2 D_{1,2}(m)}$

(3) For each $j = 1, \dots r$, $\overleftrightarrow{P_1 D_{1,3}(j)}$, $\overleftrightarrow{P_2 D_{1,3}(j)}$, and $\overleftrightarrow{P_3 D_{1,3}(j)}$

(4) For each $m \geq 1$, and for each $j = 1, \dots r$, $\overleftrightarrow{D_{1,2}(m) D_{1,3}(j)}$

(5) Any line in the form $\overleftrightarrow{D_{1,2}(t) D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(r+1) D_{1,2}(r+2)}$

(6) Any line in the form $\overleftrightarrow{D_{1,3}(t) D_{1,3}(h)}$, where $t, h \in \{1, \dots r\}$.

By the definition of the set $\mathcal{S}_{1,3}(r+1)$, we see that $\overleftrightarrow{D_{1,2}(r+1) D_{1,2}(r+2)} \notin \mathcal{S}_{1,3}(r+1)$. Let $D_{1,3}(r+1)$ be a point on $\overleftrightarrow{D_{1,2}(r+1) D_{1,2}(r+2)}$ such that

(1) $d(P_1, D_{1,3}(r+1)) \geq r+1$ and $d(P_3, D_{1,3}(r+1)) \geq r+1$

(2) $D_{1,3}(r+1)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(r+1)$

Since $D_{1,3}(r+1)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(r+1)$, then neither $n(P_1, D_{1,3}(r+1))$ nor $n(P_3, D_{1,3}(r+1))$ has yet been defined. We define $n(P_1, D_{1,3}(r+1)) = n(P_3, D_{1,3}(r+1)) = \dfrac{1}{2^{r+2}} n(P_1, P_3)$.

Thus, the arc length of the path from P_1 to P_3 along the segments $\overline{P_1 D_{1,3}(r+1)}$ and $\overline{P_3 D_{1,3}(r+1)}$ is $\dfrac{1}{2^{r+1}} n(P_1, P_3)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1 D_{1,3}(r+1)}$ or between any two points on the line $\overleftrightarrow{P_3 D_{1,3}(r+1)}$.

Also, for any $X, Y \in \overleftrightarrow{P_2 D_{1,3}(r+1)}$ we define $n(X, Y) = d(X, Y)$. Moreover, for each $m \neq r+1, r+2$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m) D_{1,3}(r+1)}$, we define $n(X, Y) = d(X, Y)$.

Thus, there exists a sequence $(D_{1,3}(j))$ of distinct points in $\mathbb{R}^2$ such that

(1) For each $j \geq 1$, $D_{1,3}(j)$ is a point on the line $\overleftrightarrow{D_{1,2}(j) D_{1,2}(j+1)}$.

(2) No three of the points in $(D_{1,3}(j))$ are collinear.

(3) For each $j \geq 1$, $D_{1,3}(j)$ is not a point on any of the following lines:

  (a) $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_1 P_3}$, and $\overleftrightarrow{P_2 P_3}$

  (b) For each $m \geq 1$, $\overleftrightarrow{P_1 D_{1,2}(m)}$ and $\overleftrightarrow{P_2 D_{1,2}(m)}$

  (c) For each $k \neq j$, $\overleftrightarrow{P_1 D_{1,3}(k)}$, $\overleftrightarrow{P_2 D_{1,3}(k)}$, and $\overleftrightarrow{P_3 D_{1,3}(k)}$

  (d) For each $m \geq 1$, and for each $k \neq j$, $\overleftrightarrow{D_{1,2}(m) D_{1,3}(k)}$

(e) Any line in the form $\overleftrightarrow{D_{1,2}(t)D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(j)D_{1,2}(j+1)}$

(f) Any line in the form $\overleftrightarrow{D_{1,3}(t)D_{1,3}(h)}$, where $t \neq j$ and $h \neq j$.

(4) For each $j \geq 1$, $d(P_1, D_{1,3}(j)) \geq j$ and $d(P_3, D_{1,3}(j)) \geq j$

(5) For each $j \geq 1$, $n(P_1, D_{1,3}(j)) = n(P_3, D_{1,3}(j)) = \dfrac{1}{2^{j+1}} n(P_1, P_3)$.

(6) For each $j \geq 1$, distance in $\mathbb{M}_1$ is defined between any two points on the line $\overleftrightarrow{P_1 D_{1,3}(j)}$ or between any two points on the line $\overleftrightarrow{P_3 D_{1,3}(j)}$ using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$.

Also, for each $j \geq 1$, and for any $X, Y \in \overleftrightarrow{P_2 D_{1,3}(j)}$ we define $n(X,Y) = d(X,Y)$. Moreover, for each $m \neq j, j+1$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m)D_{1,3}(j)}$, we define $n(X,Y) = d(X,Y)$.

Using the enumeration defined above on the set of unordered pairs of points (P_r,P_t), one can repeat the above process to construct a sequence $(D_{r,t}(m))$ of points in $\mathbb{R}^2$ for each pair of distinct points P_r and P_t in $\mathbb{Q}^2$, with $r < t$, such that for each $m \geq 1$, $d(P_r, D_{r,t}(m)) \geq m$ and $d(P_t, D_{r,t}(m)) \geq m$, and such that there exists a path from P_r to P_t passing through $D_{r,t}(m)$ whose arc length in $\mathbb{M}_1$ is equal to $\dfrac{n(P_r, P_t)}{2^m}$.

If l is a line in $\mathbb{R}^2$ that is not used in this construction, then we define the distance in $\mathbb{M}_1$ between any two points X and Y on l to be the usual euclidean distance $d(X,Y)$.

5. A Proof that $\mathbb{M}_1$ Satisfies the Ruler Postulate

In this section, we give a proof that $\mathbb{M}_1$ satisfies the Ruler Postulate. We leave it to the reader to check that in general, if a model satisfies the Ruler Postulate, then that model also satisfies the congruence axioms for segments.

Given a line l, then it follows by the way that distance is defined in $\mathbb{M}_1$ that either for each pair of points X and Y on l, $n(X,Y) = d(X,Y)$, or else there exists a segment $\overline{BH}$ on l (with $\overleftrightarrow{BH} = l$) such that:

(1) the euclidean length $d(B,H)$ of $\overline{BH}$ has been altered to define the new length $n(B,H)$ of $\overline{BH}$ in $\mathbb{M}_1$

(2) For each pair of points U and V in segment $\overline{BH}$, there exists $r \in [0,1]$ such that both $n(U,V) = rn(B,H)$ and $d(U,V) = rd(B,H)$

(3) For each pair of points X and Y on l such that $X - Y - B - H$, $n(X,Y) = d(X,Y)$

(4) For each pair of points X and Y on l such that $B - H - X - Y$, $n(X,Y) = d(X,Y)$

(5) For each point X on l such that $X - B - H$, $n(X,B) = d(X,B)$

(6) For each point Y on l such that $B - H - Y$, $n(H,Y) = d(H,Y)$

For all other possible pairs of points X and Y on l, we employ segment addition to compute $n(X,Y)$. For example, if $X - B - Y - H$, then $n(X,Y) = n(X,B) + n(B,Y)$.

First assume that for each pair of points X and Y on l, $n(X,Y) = d(X,Y)$. Since euclidean distance d satisfies the Ruler Postulate (page 31 of [8]), then there exists a bijection $f : l \to \mathbb{R}$ such that for each pair of points X and Y on l, $d(X,Y) = |f(X) - f(Y)|$. In this case, we use f as a coordinate system for l, and for each pair of points X and Y on l, we see that $n(X,Y) = d(X,Y) = |f(X) - f(Y)|$.

Next, assume that there exists a segment $\overline{BH}$ on l (with $\overleftrightarrow{BH} = l$) such that conditions (1) through (6) above are satisfied. We will define a bijection $g : l \to \mathbb{R}$ such that for each pair of points X and Y on l, $n(X,Y) = |g(X) - g(Y)|$.

Again, since euclidean distance d satisfies the Ruler Postulate, then there exists a coordinate system $f : l \to \mathbb{R}$ such that $f(B) = 0$, $f(H) > 0$, and for each point X on l such that $X - B - H$, $f(X) < 0$ [7].

Let $g(B) = 0$ and $g(H) = n(B,H)$. For each point X on l such that $X - B - H$, let $g(X) = f(X)$. Let $h = n(B,H) - d(B,H) = g(H) - f(H)$. For each point Y on l such that $B - H - Y$, let $g(Y) = f(Y) + h = f(Y) + g(H) - f(H)$. Let $t = \dfrac{n(B,H)}{d(B,H)}$. For each point X on l such that $B - X - H$, let $g(X) = tf(X)$.

We note that $g(B) = 0 = (t)(0) = tf(B)$, and that $g(H) = n(B,H) = (d(B,H))\left(\dfrac{n(B,H)}{d(B,H)}\right) = td(B,H) = t|f(B) - f(H)| = t|0 - f(H)| = tf(H)$. We also note that $g(H) = f(H) + g(H) - f(H) = f(H) + h$. We first prove that g is a bijection.

Proof that g is a bijection. Let $k \in \mathbb{R}$. First assume that $k = 0$. By definition of g, we have that $g(B) = 0$. Suppose that there exists a point X on l, with $X \neq B$ such that $g(X) = 0$. Since $g(H) > 0$, then $X \neq H$. If $X - B - H$, then $g(X) < 0$, a contradiction. If $B - X - H$, then $f(X) > 0$ and moreover, $g(X) = tf(X) > 0$, again a contradiction. If $B - H - X$, then $f(X) > f(H)$, which implies that $g(X) = f(X) + h > f(H) + h = g(H) > 0$, also a contradiction. (As above, $h = n(B,H) - d(B,H) = g(H) - f(H)$.) Thus, B is the only point such that $g(B) = 0$.

Assume that $k < 0$. Since $f : l \to \mathbb{R}$ is a bijection then there exists a unique point X_k on l such that $X_k - B - H$ and $f(X_k) = k$. By definition of g, we have that $g(X_k) = f(X_k) = k$. As just argued in the previous case, if $X = B$, $B - X - H$, $X = H$, or $B - H - X$, then $g(X) \geq 0$. Thus, X_k is the only point such that $g(X_k) = k$.

Assume that $k \geq n(B,H)$. Again, let $h = n(B,H) - d(B,H)$. Since $k \geq n(B,H)$, then $k - h \geq n(B,H) - h$. Therefore, $k - h \geq n(B,H) - (n(B,H) - d(B,H))$, which implies that $k - h \geq d(B,H)$. Since f is a coordinate system for l, then there exists a unique point Y_k on l such that $f(Y_k) = k - h$. Since $f(Y_k) = k - h \geq d(B,H)$, then either $Y_k = H$ or else $B - H - Y_k$. Thus, there exists a unique point Y_k on l such that either $Y_k = H$ or else $B - H - Y_k$, and such that $g(Y_k) = f(Y_k) + h = k$.

Finally, assume that $0 < k < n(B, H)$. Again, let $t = \dfrac{n(B,H)}{d(B,H)}$. Since f is a coordinate system for l, then there exists a unique point W_k on l such that $f(W_k) = \dfrac{k}{t}$. Since $0 < k < n(B, H)$ and since $t > 0$, then $0 = \dfrac{0}{t} < \dfrac{k}{t} < \dfrac{n(B,H)}{t} = \dfrac{n(B,H)}{\left(\dfrac{n(B,H)}{d(B,H)}\right)} = d(B, H)$. Since $f(B) = 0$, $f(H) = d(B, H)$, and $f(W_k) = \dfrac{k}{t}$, then $f(B) < f(W_k) < f(H)$, which implies that $B - W_k - H$. By definition of g, we see that $g(W_k) = tf(W_k) = (t)\left(\dfrac{k}{t}\right) = k$. Thus, there exists a unique point W_k such that $B - W_k - H$ and $g(W_k) = k$.

In any case, we see that there exists a unique point X on l such that $g(X) = k$. Hence, g is a bijection. □

We next prove that g is a coordinate system for l.

Proof that g is a coordinate system for l. We now check that for each pair of points X and Y on l, $n(X, Y) = |g(X) - g(Y)|$.

First assume that $X - Y - B - H$. In this case, we have that $n(X, Y) = d(X, Y)$, $g(X) = f(X)$, and $g(Y) = f(Y)$. Thus, $n(X, Y) = d(X, Y) = |f(X) - f(Y)| = |g(X) - g(Y)|$.

Similarly, if $X - B - H$, then we have that $n(X, B) = d(X, B)$, $g(X) = f(X)$, and $g(B) = f(B) = 0$. Thus, $n(X, B) = d(X, B) = |f(X) - f(B)| = |g(X) - g(B)|$.

Assume that $B - H - X - Y$. In this case, we have that $n(X, Y) = d(X, Y)$, $g(X) = f(X) + h$, and $g(Y) = f(Y) + h$, where, as above, $h = n(B, H) - d(B, H)$. Thus, $n(X, Y) = d(X, Y) = |f(X) - f(Y)| = |f(X) - f(Y) + h - h| = |(f(X) + h) - (f(Y) + h)| = |g(X) - g(Y)|$.

Similarly, if $B - H - Y$, then we have that $n(H, Y) = d(H, Y)$, $g(H) = f(H) + h$, and $g(Y) = f(Y) + h$. Thus, $n(H, Y) = d(H, Y) = |f(H) - f(Y)| = |f(H) - f(Y) + h - h| = |(f(H) + h) - (f(Y) + h)| = |g(H) - g(Y)|$.

Next, assume that X and Y are on segment $\overline{BH}$. As above, we let $t = \dfrac{n(B,H)}{d(B,H)}$. In particular, $n(B, H) = td(B, H)$. Also, we have that $g(X) = tf(X)$ and $g(Y) = tf(Y)$. Moreover, there exists $r \in [0, 1]$ such that both $n(X, Y) = rn(B, H)$ and $d(X, Y) = rd(B, H)$. Thus, we see that $n(X, Y) = rn(B, H) = rtd(B, H) = td(X, Y) = t|f(X) - f(Y)| = |tf(X) - tf(Y)| = |g(X) - g(Y)|$.

For any other possible combination of points X and Y on l, we use segment addition. For example, assume that $X - B - Y - H$. In this case, $g(X) < g(B) < g(Y)$, which implies that $|g(X) - g(B)| + |g(B) - g(Y)| = |g(X) - g(Y)|$. Thus, we see that $n(X, Y) = n(X, B) + n(B, Y) = |g(X) - g(B)| + |g(B) - g(Y)| = |g(X) - g(Y)|$. The cases where $B - X - H - Y$ or $X - B - H - Y$, as well as the cases where either $B - H - Y$ with $X = B$, or else $X - B - H$ with $Y = H$ are similar and are left to the reader.

Hence, g is a coordinate system for l. □

6. Convex Polygons

In this section we extend some results from previous sections to all of $\mathbb{R}^2$. In particular, we prove that given any two points $A, B \in \mathbb{R}^2$ and given any $\epsilon \in (0,1)$, then there exists a path q from A to B whose arc length in $\mathbb{M}_1$ is less than ϵ. We begin with the following lemmas.

Lemma 1. *Let $D = (a,b)$ be a point in $\mathbb{R}^2$ such that $b \notin \mathbb{Q}$. Let l denote the vertical line $x = a$. Let $\delta \in (0,1)$. Then there exists a point $T = (a,q)$ such that $q \in \mathbb{Q}$, $d(D,T) < \delta$, and $n(D,T) < \delta$.*

Proof. Given the vertical line l, then it follows by the way that distance in $\mathbb{M}_1$ is constructed above that either for each pair of points X and Y on l, $n(X,Y) = d(X,Y)$, or else there exists a segment $\overline{HB}$ (with $\overleftrightarrow{HB} = l$) such that:

(1) For each pair of points X and Y on l such that either $X - Y - H - B$ or $H - B - X - Y$, we have that $n(X,Y) = d(X,Y)$
(2) For each pair of points U and V on segment $\overline{HB}$, there exists $r \in [0,1]$ such that $n(U,V) = rn(H,B)$ and $d(U,V) = rd(H,B)$

First assume that for each pair of points X and Y on l, $n(X,Y) = d(X,Y)$. In this case, it follows by the density of the rational numbers that there exists $q_0 \in \mathbb{Q}$ such that $|b - q_0| < \delta$. In this case the point $T_0 = (a, q_0)$ is such that $n(D,T_0) = d(D,T_0) = |b - q_0| < \delta$.

Next assume that there exists a segment $\overline{HB}$ (with $\overleftrightarrow{HB} = l$) such that:

(1) For each pair of points X and Y on l such that either $X - Y - H - B$ or $H - B - X - Y$, we have that $n(X,Y) = d(X,Y)$
(2) For each pair of points U and V on segment $\overline{HB}$, there exists $r \in [0,1]$ such that $n(U,V) = rn(H,B)$ and $d(U,V) = rd(H,B)$

Let $H = (a,k)$. Since D cannot equal both H and B simultaneously, then we may assume without loss of generality that $D \neq H$. Furthermore, we may assume that $k < b$. The proof when $b < k$ is similar and is left to the reader.

First assume that D is a point on segment $\overline{HB}$. Thus, either $D = B$ or else $B - D - H$. Let $\rho \in (0,1)$ be such that $(\rho)(d(D,H)) < \delta$ and $(\rho)(n(D,H)) < \delta$. Again, by the density of the rational numbers that there exists $q_1 \in \mathbb{Q}$ such that $k < q_1 < b$ and $|b - q_1| < (\rho)(d(D,H))$. Let $T_1 = (a, q_1)$. Thus, we see that $d(D,T_1) = |b - q_1| < (\rho)(d(D,H)) < \delta$. Therefore, $n(D,T_1) < (\rho)(n(D,H)) < \delta$.

Finally, assume that $D - H - B$. Again, by the density of the rational numbers, there exists $q_2 \in \mathbb{Q}$ such that $k < q_2 < b$ and $|b - q_2| < \delta$. Let $T_2 = (a, q_2)$. Thus, we see that $n(D,T_2) = d(D,T_2) = |b - q_2| < \delta$. The case where $H - B - D$ is similar and is left to the reader. □

The proof of the following lemma is similar to the proof of Lemma 1 and is left to the reader.

Lemma 2. *Let $D = (a,b)$ be a point in $\mathbb{R}^2$ such that $a \notin \mathbb{Q}$. Let l denote the horizontal line $y = b$. Let $\delta \in (0,1)$. Then there exists a point $T = (q,b)$ such that $q \in \mathbb{Q}$, $d(D,T) < \delta$, and $n(D,T) < \delta$.*

Lemma 3. *Let $\delta \in (0,1)$. Given a point $D \in \mathbb{R}^2$, then there exists a point $T \in \mathbb{Q}^2$ and a path p from D to T such that $d(D,T) < \delta$ and such that the arc length in $\mathbb{M}_1$ along p is at most δ.*

Proof. If $D \in \mathbb{Q}^2$, then we let $T = D$. In this case p consists of the single point D and the length along p is 0.

Assume that $D \notin \mathbb{Q}^2$. Let $D = (a,b)$. If $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$, then it follows by Lemma 1 that there exists a point $T_1 = (a, q_1) \in \mathbb{Q}^2$ such that $d(D,T_1) < \delta$ and $n(D,T_1) < \delta$. In this case p is segment $\overline{DT_1}$.

If $a \notin \mathbb{Q}$ and $b \in \mathbb{Q}$, then it follows by Lemma 2 that there exists a point $T_2 = (q_2, b) \in \mathbb{Q}^2$ such that $d(D,T_2) < \delta$ and $n(D,T_2) < \delta$. In this case p is segment $\overline{DT_2}$.

Assume that $a, b \notin \mathbb{Q}$. By Lemma 1 there exists a point $W = (a, q_3)$ such that $q_3 \in \mathbb{Q}$, $d(D,W) < \frac{\delta}{2}$, and $n(D,W) < \frac{\delta}{2}$. By Lemma 2 there exists a point $T_3 = (q_4, q_3) \in \mathbb{Q}^2$ such that $d(W,T_3) < \frac{\delta}{2}$ and $n(W,T_3) < \frac{\delta}{2}$. In this case we let p be the union of segments $\overline{DW} \cup \overline{WT_3}$. Thus, the arc length along p from D to T_3 is less than δ. Moreover, by applying the triangle inequality in euclidean geometry, we have that $d(D,T_3) < d(D,W) + d(W,T_3) < \dfrac{\delta}{2} + \dfrac{\delta}{2} = \delta$. □

Theorem 4. *Given any convex polygon $\mathcal{P}$, then for each pair of points A and B inside $\mathcal{P}$ and for any $\epsilon > 0$, there exists a path q from A to B such that the arc length along q is less than ϵ and such that q has nonempty intersection with the exterior of $\mathcal{P}$.*

Proof. Let $\mathcal{P}$ be a convex polygon. Let A and B denote two points in the interior of $\mathcal{P}$. Let $\epsilon > 0$. Since $\mathcal{P}$ is a convex polygon in both $\mathbb{M}_1$ and euclidean geometry, then there exists a euclidean circle $\mathcal{C}$ that bounds $\mathcal{P}$. Let V and r denote the center and radius of $\mathcal{C}$, respectively. Let $\delta > 0$ be such that $\delta < \min\left\{\dfrac{r - d(A,V)}{2}, \dfrac{r - d(B,V)}{2}, \dfrac{\epsilon}{3}\right\}$. If $A \in \mathbb{Q}^2$, then let $W_A = A$. If $A \notin \mathbb{Q}^2$, then it follows by Lemma 3 that there exists $W_A \in \mathbb{Q}^2$ such that $d(A, W_A) < \delta$ and such that there exists a path q_A from A to W_A whose arc length in $\mathbb{M}_1$ is less than δ. Similarly, if $B \in \mathbb{Q}^2$, then let $W_B = B$, and if $B \notin \mathbb{Q}^2$, then there exists $W_B \in \mathbb{Q}^2$ such that $d(B, W_B) < \delta$ and such that there exists a path q_B from B to W_B whose arc length in $\mathbb{M}_1$ is less than δ. In either case, it follows by the triangle inequality in euclidean geometry that $d(V, W_A) < r$ and $d(V, W_B) < r$. Thus, both W_A and W_B are in the interior of $\mathcal{C}$. It also follows that the respective arc lengths of q_A and q_B in $\mathbb{M}_1$ are both less than $\frac{\epsilon}{3}$. By the construction above, there exists a sequence (D_j) of points in $\mathbb{R}^2$ such that for each $j \geq 1$,

(1) $d(W_A, D_j) \geq j$ and $d(W_B, D_j) \geq j$

(2) $n(W_A, D_j) = n(W_B, D_j) = \dfrac{1}{2^{j+1}} n(W_A, W_B)$

Let $t \in \mathbb{Z}^+$ be such that $t > 3r$ and $\dfrac{1}{2^t} n(W_A, W_B) < \dfrac{\epsilon}{3}$. Thus, D_t is a point such that

(1) $d(W_A, D_t) \geq t > 3r$ and $d(W_B, D_t) \geq t > 3r$
(2) The arc length in $\mathbb{M}_1$ from W_A to W_B along the path $p = \overline{W_A D_t} \cup \overline{W_B D_t}$ is less than $\frac{1}{2^t} n(W_A, W_B)$, and therefore less than $\frac{\epsilon}{3}$.

Thus we see that the arc length in $\mathbb{M}_1$ from A to B along the path $q = q_A \cup p \cup q_B$ is less than ϵ.

If $d(V, D_t) \leq r$, then it follows by the triangle inequality in euclidean geometry that $d(W_A, D_t) < d(W_A, V) + d(V, D_t) < r + r = 2r$, a contradiction. Therefore, it must be the case that D_t is in the exterior of $\mathcal{C}$, and therefore in the exterior of $\mathcal{P}$. Since D_t is a point on the path q, then q has nonempty intersection with the exterior of $\mathcal{P}$. □

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