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A Model of Nowhere Geodesic Plane Geometry in which the Triangle Inequality Fails Everywhere

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Abstract. Continuing with the results from an earlier paper, we construct a model $\mathbb{M}_2$ of plane geometry that satisfies all of Hilbert's axioms for the euclidean plane (with the exception of Sided-Angle-Side), yet in which the geodesic line segment connecting any two points A and B is never the shortest path from A to B. Moreover, in the model $\mathbb{M}_2$, the triangle inequality always fails for any triple of noncollinear points.

1. Introduction

In [2] the author constructs a model $\mathbb{M}_1$ of plane geometry that satisfies all of Hilbert's axioms for the euclidean plane (with the exception of Sided-Angle-Side), yet in which the geodesic line segment connecting any two points A and B is never the shortest path from A to B. Moreover, it is shown in [2] that the model $\mathbb{M}_1$ is continuous in the sense that it satisfies both the ruler postulate and protractor postulate of Birkhoff [1], [4]. However, depending on the order in which we choose points while constructing $\mathbb{M}_1$, there might exist a specific triangle $\triangle ABC$ in $\mathbb{M}_1$ that satisfies the triangle inequality in the sense that the sum of the lengths of any two sides of $\triangle ABC$ is greater than the length of the third side. However, given any two of the vertices of $\triangle ABC$, and given $\epsilon > 0$, then we can always find a path connecting those two vertices whose arc length in $\mathbb{M}_1$ is less than ϵ.

In this paper, we construct a model $\mathbb{M}_2$ of plane geometry that satisfies all of Hilbert's axioms for the euclidean plane (with the exception of Sided-Angle-Side), and which is Nowhere Geodesic, yet in which no triangle satisfies the triangle inequality. We note that the model $\mathbb{M}_2$ is not continuous in the sense that neither the ruler postulate nor the protractor postulate hold in $\mathbb{M}_2$. We refer the reader to [2] and [3] for a list of Hilbert's axioms for the euclidean plane.

2. The Model $\mathbb{M}_2$

In this section we construct the model $\mathbb{M}_2$. The points in $\mathbb{M}_2$ are precisely the points in $\mathbb{Q}^2$. A line in $\mathbb{M}_2$ is the intersection of $\mathbb{Q}^2$ with a line in $\mathbb{R}^2$ whose equation is either $y = mx + b$, where $m, b \in \mathbb{Q}$, or else $x = k$, where $k \in \mathbb{Q}$. We define betweenness in $\mathbb{M}_2$ to be exactly the same as betweenness in the euclidean

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plane. As in [2], we will replace PASCH with PSP as a betweenness axiom, and will show that $\mathbb{M}_2$ satisfies PSP. It follows immediately that betweenness axioms (1) - (3) hold in $\mathbb{M}_2$. We leave it to the reader to check that the incidence axioms hold in $\mathbb{M}_2$.

We prove that PSP holds in $\mathbb{M}_2$. We will essentially prove part (3) of the following lemma while showing that PSP holds in $\mathbb{M}_2$.

Lemma 1. *Given two distinct lines l and q in $\mathbb{M}_2$, then exactly one of the following is true:*

(1) *neither l nor q is vertical, and they have the same slope and are therefore parallel*
(2) *l and q are both vertical, and are therefore parallel*
(3) *l and q intersect at a unique point $P \in \mathbb{Q}^2$*

Proof of PSP: Let l be a line in $\mathbb{M}_2$. First assume that l is not vertical, and that l has equation $y = mx + b$, where $m, b \in \mathbb{Q}$. Let $\mathcal{H}_1 = \{(c, d) \in \mathbb{Q}^2 \mid d > mc + b\}$ and let $\mathcal{H}_2 = \{(c, d) \in \mathbb{Q}^2 \mid d < mc + b\}$. It follows immediately that $\mathcal{H}_1$ and $\mathcal{H}_2$ are nonempty, disjoint subsets of $\mathbb{Q}^2$ whose union is precisely the set of points in $\mathbb{Q}^2$ that are not on l. Let $B_1 = (u_1, v_1)$ and $B_2 = (u_2, v_2)$ be two points in $\mathbb{Q}^2$ such that B_1 and B_2 are not on l.

Assume that $\overleftrightarrow{B_1B_2}$ is not vertical. Thus, $u_1 \neq u_2$. Let $\overleftrightarrow{B_1B_2}$ have equation $y = rx + s$, where $r = \dfrac{v_2 - v_1}{u_2 - u_1} \in \mathbb{Q}$ and $s \in \mathbb{Q}$.

First assume that $B_1 \in \mathcal{H}_1$ and $B_2 \in \mathcal{H}_2$. Since $r, s, m, b \in \mathbb{Q}$, then both of $\dfrac{s-b}{m-r}$ and $r\left(\dfrac{s-b}{m-r}\right) + s$ are in $\mathbb{Q}$. Thus, we see that the lines $\overleftrightarrow{B_1B_2}$ and l intersect at the point

$$\left(\frac{s-b}{m-r}, r\left(\frac{s-b}{m-r}\right) + s\right) \in \mathbb{Q}^2$$

If $m = r$, then $\overleftrightarrow{B_1B_2}$ and l are parallel, which implies that either both of B_1 and B_2 are above l or else both of B_1 and B_2 are below l, a contradiction. Assume that $m \neq r$, and consequently, that $\overleftrightarrow{B_1B_2}$ and l are not parallel.

Assume that $m > r$. Since $B_1 \in \mathcal{H}_1$, then $ru_1 + s = v_1 > mu_1 + b$. This implies that $(m-r)(u_1) < s - b$, and therefore that $u_1 < \dfrac{s-b}{m-r}$. Since $B_2 \in \mathcal{H}_2$, then $ru_2 + s = v_2 < mu_2 + b$. This implies that $(m-r)(u_2) > s - b$, and therefore that $u_2 > \dfrac{s-b}{m-r}$. Thus, $u_1 < \dfrac{s-b}{m-r} < u_2$. Since $\dfrac{s-b}{m-r}$ is between u_1 and u_2, and since $y = rx + s$ is linear, then $\left(\dfrac{s-b}{m-r}, r\left(\dfrac{s-b}{m-r}\right) + s\right)$ is between B_1 and B_2 on the segment $\overline{B_1B_2}$.

Now assume that $m < r$. Again, since $B_1 \in \mathcal{H}_1$, then $ru_1 + s = v_1 > mu_1 + b$. This implies that $(m-r)(u_1) < s - b$, and therefore that $u_1 > \dfrac{s-b}{m-r}$. Since $B_2 \in \mathcal{H}_2$, then $ru_2 + s = v_2 < mu_2 + b$. This implies that $(m-r)(u_2) > s - b$.

and therefore that $u_2 < \dfrac{s-b}{m-r}$. Thus, $u_1 > \dfrac{s-b}{m-r} > u_2$. Thus, we again have that $\left(\dfrac{s-b}{m-r}, r\left(\dfrac{s-b}{m-r}\right)+s\right)$ is between B_1 and B_2 on the segment $\overline{B_1B_2}$.

Now assume that B_1 and B_2 are both in $\mathcal{H}_1$. Since $u_1 \neq u_2$, then we may assume without loss of generality that $u_1 < u_2$. The proof for the case $u_1 > u_2$ is similar and is left to the reader.

Again, if $m = r$, then $\overleftrightarrow{B_1B_2}$ and l are parallel, which implies that $\overleftrightarrow{B_1B_2}$ does not intersect l.

Assume that $m > r$. Since $B_2 \in \mathcal{H}_1$, then $ru_2 + s = v_2 > mu_2 + b$. This implies that $(m-r)(u_2) < s-b$, and therefore that $u_1 < u_2 < \dfrac{s-b}{m-r}$.

Assume that $m < r$. Since $B_1 \in \mathcal{H}_1$, then $ru_1 + s = v_1 > mu_1 + b$. This implies that $(m-r)(u_1) < s-b$, and therefore that $u_2 > u_1 > \dfrac{s-b}{m-r}$.

In either case, since $y = rx + s$ is linear, then segment $\overline{B_1B_2}$ does not intersect l.

If B_1 and B_2 are both in $\mathcal{H}_2$, then one can use a similar argument to show that segment $\overline{B_1B_2}$ does not intersect l.

Now assume that $\overleftrightarrow{B_1B_2}$ is vertical. Thus, $u_1 = u_2$, and $\overleftrightarrow{B_1B_2}$ has equation $x = u_1$.

Assume that $B_1 \in \mathcal{H}_1$ and $B_2 \in \mathcal{H}_2$. In this case, the lines $\overleftrightarrow{B_1B_2}$ and l intersect at the point $(u_1, mu_1 + b) \in \mathbb{Q}^2$. Since $B_1 \in \mathcal{H}_1$ and $B_2 \in \mathcal{H}_2$, then $v_1 > mu_1 + b > v_2$. Thus, it follows that $(u_1, mu_1 + b)$ is between that points B_1 and B_2 on the segment $\overline{B_1B_2}$.

Assume that $B_1, B_2 \in \mathcal{H}_1$. We may assume without loss of generality that $v_1 > v_2$. Again, the lines $\overleftrightarrow{B_1B_2}$ and l intersect at the point $(u_1, mu_1 + b) \in \mathbb{Q}^2$. Since $B_1, B_2 \in \mathcal{H}_1$ and $v_1 > v_2$, then $v_1 > v_2 > mu_1 + b$. Thus, the point $(u_1, mu_1 + b)$ is not between B_1 and B_2. If B_1 and B_2 are both in $\mathcal{H}_2$, then one can use a similar argument to show that segment $\overline{B_1B_2}$ does not intersect l.

Now assume that l is vertical, and that l has equation $x = k$, where $k \in \mathbb{Q}$. Let $\mathcal{H}_3 = \{(c,d) \in \mathbb{Q}^2 \mid c > k\}$ and let $\mathcal{H}_4 = \{(c,d) \in \mathbb{Q}^2 \mid c < k\}$. Again, we leave it to the reader to confirm that $\mathcal{H}_3$ and $\mathcal{H}_4$ are nonempty, disjoint subsets of $\mathbb{Q}^2$ whose union is precisely the set of points in $\mathbb{Q}^2$ that are not on l. Let $B_3 = (u_3, v_3)$ and $B_4 = (u_4, v_4)$ be two points in $\mathbb{Q}^2$ such that B_3 and B_4 are not on l.

Assume that $B_3 \in \mathcal{H}_3$ and $B_4 \in \mathcal{H}_4$. In this case, $\overleftrightarrow{B_3B_4}$ can not be vertical. Let $\overleftrightarrow{B_3B_4}$ have equation $y = px + d$, where $p, d \in \mathbb{Q}$. In this case, the lines $\overleftrightarrow{B_3B_4}$ and l intersect at the point $(k, pk + d) \in \mathbb{Q}^2$. Since $B_3 \in \mathcal{H}_3$ and $B_4 \in \mathcal{H}_4$, then $u_3 > k > u_4$. Thus, it follows that $(k, pk + d)$ is between that points B_3 and B_4 on the segment $\overline{B_3B_4}$.

Finally, assume that $B_3, B_4 \in \mathcal{H}_3$. If $\overleftrightarrow{B_3B_4}$ is vertical, then segment $\overline{B_3B_4}$ and line l do not intersect. Assume that $\overleftrightarrow{B_3B_4}$ is not vertical. Again, we assume that $\overleftrightarrow{B_3B_4}$ has equation $y = px + d$, where $p, d \in \mathbb{Q}$. We may assume without loss of generality that $u_3 > u_4$. Since $B_3, B_4 \in \mathcal{H}_3$, then $u_3 > u_4 > k$. Thus, it follows that $(k, pk+d)$ is not between that points B_3 and B_4 on the segment $\overline{B_3B_4}$. If B_3 and B_4 are both in $\mathcal{H}_4$, then one can use a similar argument to show that segment $\overline{B_3B_4}$ does not intersect l. Hence, PSP holds in $\mathbb{M}_2$.

3. Angle Measure in $\mathbb{M}_2$

We define angle measure in $\mathbb{M}_2$ to be the same as euclidean angle measure. It follows immediately that Congruence Axiom (2) for Angles holds in $\mathbb{M}_2$. Moreover, given an angle $\angle BAC$ in $\mathbb{M}_2$ and given a ray $\overrightarrow{DF}$ in $\mathbb{M}_2$, then there exists a unique ray $\overrightarrow{DE}$ in $\mathbb{R}^2$ on a given side of line $\overleftrightarrow{DF}$ such that $\angle BAC \cong \angle FDE$. It remains only to show that ray $\overrightarrow{DE}$ is not only a ray in $\mathbb{R}^2$, but more specifically is a ray in $\mathbb{M}_2$. To show this, we show that either $\overleftrightarrow{DE}$ is vertical or else that the slope of line $\overleftrightarrow{DE}$ is a rational number.

Since both the slope of a line and euclidean angle measure are preserved under horizontal or vertical translation, then we may assume that D is the origin (0,0). Similarly, we may assume that the vertex A of $\angle BAC$ is the origin (0,0).

If the slope of a ray $\overrightarrow{DK}$ is $s \in \mathbb{Q}$, then the slope of the ray that we get by rotating $\overrightarrow{DK}$ clockwise around D through an angle of $\dfrac{\pi}{2}$ is $-\dfrac{1}{s} \in \mathbb{Q}$. Thus, we can assume that ray $\overrightarrow{DF}$ is either in the first quadrant or is equal to the positive x-axis. If $\overrightarrow{DF}$ is in the second, third, or fourth quadrant, then we rotate both rays $\overrightarrow{DF}$ and $\overrightarrow{DE}$ clockwise around D through an angle of $\dfrac{\pi}{2}$, π, or $\dfrac{3\pi}{2}$, respectively. In any of these three cases, the ray we get by rotating $\overrightarrow{DF}$ will be a ray in the first quadrant with a rational slope. Moreover, if the slope of the ray we get by rotating $\overrightarrow{DE}$ is a rational number, then the slope of $\overrightarrow{DE}$ is also a rational number. Since angle measure in euclidean geometry is preserved by rotation, then the angle measure of the new angle is precisely the same as the angle measure of the original angle $\angle FDE$. Similarly, if $\overrightarrow{DF}$ is the positive y-axis, the negative x-axis, or the negative y-axis, then we again rotate both rays $\overrightarrow{DF}$ and $\overrightarrow{DE}$ clockwise around D through an angle of $\dfrac{\pi}{2}$, π, or $\dfrac{3\pi}{2}$, respectively. In any of these three cases, the ray we get by rotating $\overrightarrow{DF}$ will be the positive x-axis.

Finally, we assume that $\overrightarrow{DE}$ is in the halfplane of line $\overleftrightarrow{DF}$ consisting of all points that are above the line $\overleftrightarrow{DF}$. The case where the ray $\overrightarrow{DE}$ is in the halfplane of line $\overleftrightarrow{DF}$ consisting of all points that are below the line $\overleftrightarrow{DF}$ is similar, and is left to the reader.

Let $\angle BAC$ and $\angle FDE$ be two angles in $\mathbb{R}^2$ that are congruent. As above, we assume that the vertices A and D of angles $\angle BAC$ and $\angle FDE$, respectively, are the origin (0,0). Moreover, by rotating $\overrightarrow{AB}$ if necessary, we may assume that $\overrightarrow{AB}$ is the positive x-axis. By using the reflection in the x-axis and reflecting $\overrightarrow{AC}$ to its mirror image $\overrightarrow{AC'}$, if necessary, then we may assume that $\overrightarrow{AC}$ is above the x-axis. In particular, if the slope of $\overrightarrow{AC}$ is $s \in \mathbb{Q}$, then the slope of $\overrightarrow{AC'}$ is

$-s \in \mathbb{Q}$. Also, since the reflection in a line preserves euclidean angle measure, then $m\angle BAC = m\angle BAC'$. We denote the common angle measure of $\angle BAC$ and $\angle FDE$ by λ. We assume that ray $\overrightarrow{DF}$ has rational slope, and is therefore a ray in $\mathbb{M}_2$. We will show that ray $\overrightarrow{AC}$ is in $\mathbb{M}_2$ if and only if ray $\overrightarrow{DE}$ is in $\mathbb{M}_2$.

If $\lambda = \frac{\pi}{2}$, then $\overrightarrow{AC}$ is vertical, and the slope of $\overrightarrow{DE}$ is the negative reciprocal of the slope of $\overrightarrow{DF}$, which implies that the slope of $\overrightarrow{DE}$ is rational. In this case, $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are both in $\mathbb{M}_2$.

Assume that $\lambda \neq \frac{\pi}{2}$.

If $\overrightarrow{DF}$ is horizontal (and therefore the positive x-axis), then $\angle BAC$ and $\angle FDE$ are the same angle. In this case, $\overrightarrow{DE}$ and $\overrightarrow{AC}$ are the same ray, and therefore have the same rational slope.

Assume that $\overrightarrow{DF}$ is not horizontal. As above, we may assume that $\overrightarrow{DF}$ is in the first quadrant, and that $\overrightarrow{DE}$ is in the halfplane of line $\overleftrightarrow{DF}$ consisting of all points that are above the line $\overleftrightarrow{DF}$.

Let $\rho = m\angle BDF$.

First assume that $\overrightarrow{AC}$ is in $\mathbb{M}_2$. Since $\lambda \neq \frac{\pi}{2}$, then $\overrightarrow{AC}$ is not vertical. Thus, it must be the case that the slope of $\overrightarrow{AC}$ is rational.

If $\overrightarrow{DE}$ is vertical (i.e. the positive y-axis), then it has no slope, but instead has equation $x = 0$, and is therefore a ray in $\mathbb{M}_2$.

If $\overrightarrow{DE}$ is horizontal (i.e. the negative x-axis), then it has slope 0, and is therefore a ray in $\mathbb{M}_2$.

Assume that $\overrightarrow{DE}$ is neither the positive y-axis nor the negative x-axis. Since $\overrightarrow{AB}$ is the positive x-axis, then the slope of $\overrightarrow{AC}$ is Tan(λ), the slope of $\overrightarrow{DF}$ is Tan(ρ), and the slope of $\overrightarrow{DE}$ is Tan(λ + ρ). Since the slopes of $\overrightarrow{AC}$ and $\overrightarrow{DF}$ are rational, then it follows by the formula

$$\mathrm{Tan}(\lambda + \rho) = \frac{\mathrm{Tan}(\lambda) + \mathrm{Tan}(\rho)}{1 - \mathrm{Tan}(\lambda)\mathrm{Tan}(\rho)}$$

that Tan$(\lambda + \rho)$ is rational. Thus, the slope of $\overrightarrow{DE}$ is rational.

Thus, in any of these cases, we have that $\overrightarrow{DE}$ is a ray in $\mathbb{M}_2$.

Next, assume that $\overrightarrow{DE}$ is in $\mathbb{M}_2$.

First assume that $\overrightarrow{DE}$ is vertical (i.e. the positive y-axis). In this case, we use the reflection in the line $y = x$ to reflect both $\overrightarrow{DF}$ and $\overrightarrow{DE}$ to their mirror images $\overrightarrow{DF'}$ and $\overrightarrow{DE'}$, respectively. Since the ray $\overrightarrow{DE}$ is the positive y-axis, then $\overrightarrow{DE'}$= $\overrightarrow{AB}$. Since the reflection in a line preserves angle measure, then the reflection of the angle $\angle FDE$ is the unique angle above the x-axis whose initial ray is $\overrightarrow{AB}$ and whose angle measure is λ. But this is precisely the angle $\angle BAC$. Thus, we see that $\angle F'DE'$ = $\angle BAC$. More specifically, $\overrightarrow{DE'}$ = $\overrightarrow{AB}$ and $\overrightarrow{DF'}$= $\overrightarrow{AC}$. If the slope of $\overrightarrow{DF}$ is $s \in \mathbb{Q}$, then the slope of $\overrightarrow{DF'}$ is $\dfrac{1}{s} \in \mathbb{Q}$. Since $\overrightarrow{DF'} = \overrightarrow{AC}$, then it follows that $\overrightarrow{AC}$ is in $\mathbb{M}_2$.

Now assume that that $\overrightarrow{DE}$ is not vertical. Thus, in this case the slope of $\overrightarrow{DE}$ is rational. Moreover, since $\overrightarrow{DE}$ is not vertical, then $\lambda + \rho \neq \dfrac{\pi}{2}$.

As above, we have that the slope of $\overrightarrow{AC}$ is Tan(λ), the slope of $\overrightarrow{DF}$ is Tan(ρ), and the slope of $\overrightarrow{DE}$ is Tan(λ + ρ). Since the slopes of $\overrightarrow{DF}$ and $\overrightarrow{DE}$ are rational, then it follows by the formula

$$\text{Tan}(\lambda) = \frac{\text{Tan}(\lambda + \rho) - \text{Tan}(\rho)}{1 + \text{Tan}(\lambda + \rho)\text{Tan}(\rho)}$$

that the slope of $\overrightarrow{AC}$ is also rational. Thus, $\overrightarrow{AC}$ is a ray in $\mathbb{M}_2$.

Let $\angle HKG$ be an angle in $\mathbb{M}_2$, and let $\overrightarrow{DF}$ be a ray in $\mathbb{M}_2$. As stated above, there exists a unique ray $\overrightarrow{DE}$ in $\mathbb{R}^2$ on a given side of line $\overleftrightarrow{DF}$ such that $\angle HKG \cong \angle FDE$. If, as above, we let $\overrightarrow{AB}$ denote the positive x-axis, then there exists a unique ray $\overrightarrow{AC}$ in $\mathbb{R}^2$ above the x-axis such that $\angle BAC \cong \angle HKG$. Since $\angle HKG$ is an angle in $\mathbb{M}_2$, then it follows that both rays $\overrightarrow{KH}$ and $\overrightarrow{KG}$ are in $\mathbb{M}_2$. Applying the previous argument to the angles $\angle BAC$ and $\angle HKG$, it follows that $\overrightarrow{AC}$ is a ray in $\mathbb{M}_2$. Applying the previous argument to the angles $\angle BAC$ and $\angle FDE$, we see that since $\overrightarrow{AC}$ is a ray in $\mathbb{M}_2$, then $\overrightarrow{DE}$ is a ray in $\mathbb{M}_2$. Thus, $\angle FDE$ is an angle in $\mathbb{M}_2$.

Hence, it follows that Congruence Axiom (i) for Angles holds in $\mathbb{M}_2$.

4. Distance in $\mathbb{M}_2$

In this section we define distance in $\mathbb{M}_2$. We do this in such a way so that not only is $\mathbb{M}_2$ nowhere geodesic, but moreover, so that no triangle in $\mathbb{M}_2$ satisfies the triangle inequality. To define distance, we use the enumeration on the points in $\mathbb{Q}^2$. In particular, we start by defining distance between P_1 and P_2. We then define the distance between each subsequent point P_k in the sequence (P_j) and all of the points $P_1, \ldots, P_{k-1}$ that come before P_k in (P_j). As we define these distances, we also construct a subsequence $(K_{i,j})$ of points from (P_j), which is used to ensure that $\mathbb{M}_2$ is nowhere geodesic. The distance between a point P_t in (P_j) and a corresponding point $K_{t,j}$ in $(K_{i,j})$ is defined to be sufficiently small so that the segments in the form $\overline{P_t K_{t,i}}$ can be used to construct shorter and shorter paths in $\mathbb{M}_2$. On the other hand, certain distances in $\mathbb{M}_2$ are defined to be larger and larger as we use points further out in the sequence (P_j). Thus, we can think of distances in $\mathbb{M}_2$ as being defined with two opposite goals in mind. One goal is to have certain distances get larger and larger without bound, and the other goal is to have certain distances getting smaller and smaller and closer to 0. We denote the euclidean distance between point A and point B by $d(A, B)$. We denote the taxicab distance between point A and point B by $t(A, B)$ We denote the distance in $\mathbb{M}_2$ between point A and point B by $g(A, B)$.

We start with the following lemma.

Lemma 2. *Let $\{A_1, \ldots, A_k\} \subseteq \mathbb{Q}^2$ be a nonempty finite subset of $\mathbb{Q}^2$. Let P_i and P_l be two distinct points in $\mathbb{Q}^2$. Assume that $T_0, T_1, \ldots, T_m$ are points on $\overline{P_iP_l}$ such that*

(1) *$T_0 = P_i$ and $T_m = P_l$*
(2) *If $m \geq 2$, then for each $d \in \{1, \ldots, m-1\}$, we have that $T_{d-1} - T_d - T_{d+1}$*
(3) *If $m \geq 2$, then for each $d \in \{1, \ldots, m-1\}$, we have that T_d is the point of intersection of $\overline{P_iP_l}$ with one of the lines $\overleftrightarrow{A_hA_b}$ where $1 \leq h, b \leq k$ and $\overleftrightarrow{A_hA_b} \neq \overleftrightarrow{P_iP_l}$*
(4) *For each $d \in \{1, \ldots, m\}$, none of the lines $\overleftrightarrow{A_hA_b}$, where $1 \leq h, b \leq k$ and $\overleftrightarrow{A_hA_b} \neq \overleftrightarrow{P_iP_l}$, intersect int$\left(\overline{T_{d-1}T_d}\right)$*

Then for each $d \in \{1, \ldots, m\}$, there exists a triangle $\triangle T_{d-1}T_dQ_d$ such that

(1) *No line in the form $\overleftrightarrow{A_hA_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_{d-1}T_dQ_d)$.*
(2) *No segment in the form $\overline{A_hA_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_{d-1}T_dQ_d)$.*

Proof. We will prove this for $d = 1$. The proofs for the other possible values of d are similar, and are left to the reader. Let $\mathcal{H}$ denote a halfplane of $\overleftrightarrow{P_iP_l}$. If none of the points $A_1, \ldots, A_k$ are in $\mathcal{H}$, then we let X be any point in $\mathcal{H} \cap \mathbb{Q}^2$. Even though none of the points $A_1, \ldots, A_k$ are in $\mathcal{H}$, it might still be the case that one of the lines $\overleftrightarrow{A_hA_b}$ intersects int$(\angle T_1T_0X)$.

First assume that no line in the form $\overleftrightarrow{A_zA_w}$, where $1 \leq z, w \leq k$, passes through T_0 and intersects int$(\angle T_1T_0X)$.

Assume that no line in the form $\overleftrightarrow{A_hA_b}$, where $1 \leq h, b \leq k$, intersects int$\left(\overrightarrow{T_0X}\right)$. Let Q_1 be any point on ray $\overrightarrow{T_0X}$ such that $Q_1 \neq T_0$. If there exists a line in the form $\overleftrightarrow{A_hA_b}$, where $1 \leq h, b \leq k$, that intersects int$(\triangle T_0T_1Q_1)$, then it follows by the Line-Triangle Theorem together with Lemma 1 that $\overleftrightarrow{A_hA_b}$ must intersect the interior of at least one of the sides $\overline{T_0T_1}$ or $\overline{T_0Q_1}$, a contradiction [4]. Thus, no line in the form $\overleftrightarrow{A_hA_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_0T_1Q_1)$. In particular, no segment in the form $\overline{A_hA_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_0T_1Q_1)$.

Next assume that there exists a line in the form $\overleftrightarrow{A_hA_b}$, where $1 \leq h, b \leq k$, that intersects int$\left(\overrightarrow{T_0X}\right)$. Let $L_1, \ldots, L_r$ denote all the points of intersection of int$\left(\overrightarrow{T_0X}\right)$ with lines in the form $\overleftrightarrow{A_hA_b}$, where $1 \leq h, b \leq k$. Since there exists only a finite number of lines in the form $\overleftrightarrow{A_hA_b}$, where $1 \leq h, b \leq k$, then there exist only a finite number of such points $L_1, \ldots, L_r$ of intersection. Let L_u denote the point on int$\left(\overrightarrow{T_0X}\right)$ such that none of the other points from $L_1, \ldots, L_r$ are between T_0 and L_u. That is, for each $j \neq u$, L_j is not between T_0 and L_u. Since there exist only a finite number of points $L_1, \ldots, L_r$ of intersection, then such a point L_u exists. One can now use an argument similar to the one given above to show that no line in the form $\overleftrightarrow{A_hA_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_1T_0L_u)$.

Now assume that there exists a line in the form $\overleftrightarrow{A_zA_w}$, where $1 \leq z, w \leq k$, that passes through T_0 and intersects int$(\angle T_1T_0X)$. Since none of the points

$A_1, \ldots, A_k$ are in $\mathcal{H}$, then it follows that all points on $\overleftrightarrow{A_z A_w}$ in $\text{int}(\angle T_1 T_0 X)$ are on the opposite side of $\overleftrightarrow{P_i P_l}$ as $A_1, \ldots, A_k$. Let $\mathcal{S}$ denote the set of all angles in the form $\angle T_1 T_0 B$ such that B is in $\text{int}(\angle T_1 T_0 X)$ and such that B is on a line $\overleftrightarrow{A_z A_w}$, where $1 \leq z, w \leq k$, that passes through T_0. Let D be a point in $\text{int}(\angle T_1 T_0 X)$ such that D is on a line $\overleftrightarrow{A_z A_w}$, where $1 \leq z, w \leq k$, that passes through T_0 and such that $\angle T_1 T_0 D$ is the smallest angle in $\mathcal{S}$. We now apply arguments similar to those given above, using ray $\overrightarrow{T_0 D}$ in the place of ray $\overrightarrow{T_0 X}$, to show that triangle $\triangle T_1 T_0 D$ is such that no line in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$ and $\overleftrightarrow{A_h A_b} \neq \overleftrightarrow{P_i P_l}$, intersects $\text{int}(\triangle T_1 T_0 D)$.

Now assume that at least one of the points $A_1, \ldots, A_k$ is in $\mathcal{H}$. Let α denote the smallest angle in the form $\angle T_1 T_0 A_t$, where A_t is from the points $A_1, \ldots, A_k$ that are in $\mathcal{H}$. Since there are only a finite number of points from $A_1, \ldots, A_k$ that are in $\mathcal{H}$, then α is well defined. We now apply arguments similar to those given above, using ray $\overrightarrow{T_0 A_t}$ in the place of ray $\overrightarrow{T_0 X}$, to show that triangle $\triangle T_1 T_0 A_t$ is such that no line in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$, intersects $\text{int}(\triangle T_1 T_0 A_t)$. $\square$

The proof of the following lemma follows from the definitions of betweenness in euclidean geometry, taxicab geometry, and $\mathbb{M}_2$, and is left to the reader.

Lemma 3. *Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$ be three points in $\mathbb{Q}^2$. Then $A - B - C$ in euclidean geometry if and only if $A - B - C$ in taxicab geometry. Moreover, $A - B - C$ in taxicab geometry if and only if $A - B - C$ in $\mathbb{M}_2$.*

The proof of the following lemma follows from the definition of distance in taxicab geometry, and is left to the reader.

Lemma 4. *Let $A, B \in \mathbb{Q}^2$. Then $t(A, B) \in \mathbb{Q}$.*

Define $g(P_1, P_2) = 1$. Let $K_{1,2}$ be any point in $\mathbb{Q}^2$ such that $K_{1,2}$ is not on $\overleftrightarrow{P_1 P_2}$. Define $g(P_1, K_{1,2}) = g(P_2, K_{1,2}) = \dfrac{1}{4} g(P_1, P_2) = \dfrac{1}{4}$. Note that the arc length in $\mathbb{M}_2$ of the path from P_1 to P_2 along $\overline{P_1, K_{1,2}}$ and $\overline{P_2, K_{1,2}}$ is $\dfrac{1}{2} g(P_1, P_2)$.

First assume that $P_3 = K_{1,2}$. In this case the distances $g(P_1, P_3)$ and $g(P_2, P_3)$ are already defined. In particular, we have from above that $g(P_1, P_3) = g(P_2, P_3)$ $= \dfrac{1}{4} g(P_1, P_2) = \dfrac{1}{4}$.

By Lemma 2, there exists a triangle $\triangle P_1 P_3 Q_0$ such that neither of the lines $\overleftrightarrow{P_1 P_2}$ or $\overleftrightarrow{P_2 K_{1,2}}$ intersect the interior of $\triangle P_1 P_3 Q_0$. Let $K_{1,3}$ be a point in $\text{int}(\triangle P_1 P_3 Q_0) \cap \mathbb{Q}^2$. Note that $K_{1,3}$ is not on any of the lines $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_1 K_{1,2}}$, or $\overleftrightarrow{P_2 K_{1,2}}$. Therefore, neither $\overline{P_1 K_{1,3}}$ nor $\overline{P_3 K_{1,3}}$ are on any of the lines $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_1 K_{1,2}}$, or $\overleftrightarrow{P_2 K_{1,2}}$. We define $g(P_1, K_{1,3}) = g(P_3, K_{1,3}) = \dfrac{1}{4} g(P_1, P_3) = \dfrac{1}{16}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1 K_{1,3}}$ and $\overline{P_3 K_{1,3}}$ is $\dfrac{1}{2} g(P_1, P_3) = \dfrac{1}{8}$.

Similarly, there exists a triangle $\triangle P_2P_3\hat{Q}_0$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$ intersect the interior of $\triangle P_2P_3\hat{Q}_0$. Let $K_{2,3}$ be a point in int$(\triangle P_2P_3\hat{Q}_0) \cap \mathbb{Q}^2$. Note that $K_{2,3}$ is not on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$. Therefore, neither $\overline{P_2K_{2,3}}$ nor $\overline{P_3K_{2,3}}$ are on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$. We define $g(P_2, K_{2,3}) = g(P_3, K_{2,3}) = \dfrac{1}{4}g(P_2, P_3) = \dfrac{1}{16}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{2,3}}$ and $\overline{P_3K_{2,3}}$ is $\dfrac{1}{2}g(P_2, P_3) = \dfrac{1}{8}$.

Note that each of the points $K_{i,j}$ and $K_{i,j,t}$ defined below will at some step in the process be realized as one of the points P_i in the sequence (P_j) of points in $\mathbb{Q}^2$.

Now assume that $P_3 \neq K_{1,2}$. We have several cases when defining distance in $\mathbb{M}_2$ between P_3 and either of P_1 or P_2. First assume that P_3 is a point on $\overleftrightarrow{P_1P_2}$. We have three cases: $P_1 - P_3 - P_2$, $P_3 - P_1 - P_2$, or $P_1 - P_2 - P_3$.

Assume that $P_1 - P_3 - P_2$. In this case, since $P_1 - P_3 - P_2$ in both $\mathbb{M}_2$ and in taxicab geometry, then there exist $r_1, r_2 \in (0, 1) \cap \mathbb{Q}$ such that

(1) $r_1 + r_2 = 1$
(2) $t(P_1, P_3) = r_1 t(P_1, P_2)$
(3) $t(P_3, P_2) = r_2 t(P_1, P_2)$

Note that since $P_1 - P_3 - P_2$, then $t(P_1, P_2) = t(P_1, P_3) + t(P_3, P_2)$, which is consistent with $r_1 + r_2 = 1$.

Define $g(P_1, P_3) = r_1 g(P_1, P_2) = r_1$ and $g(P_3, P_2) = r_2 g(P_1, P_2) = r_2$.

By Lemma 2, there exists a triangle $\triangle P_1P_3Q_1$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_1$. Let $K_{1,3}$ be a point in int$(\triangle P_1P_3Q_1) \cap \mathbb{Q}^2$. Note that $K_{1,3}$ is not on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$. Therefore, neither $\overline{P_1K_{1,3}}$ nor $\overline{P_3K_{1,3}}$ are on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$. We define $g(P_1, K_{1,3}) = g(P_3, K_{1,3}) = \dfrac{1}{4}g(P_1, P_3) = \dfrac{r_1}{4}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3}}$ and $\overline{P_3K_{1,3}}$ is $\dfrac{1}{2}g(P_1, P_3)$.

Similarly, there exists a triangle $\triangle P_2P_3Q_2$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$ intersect the interior of $\triangle P_2P_3Q_2$. Let $K_{2,3}$ be a point in int$(\triangle P_2P_3Q_2) \cap \mathbb{Q}^2$. Note that $K_{2,3}$ is not on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$. Therefore, neither $\overline{P_2K_{2,3}}$ nor $\overline{P_3K_{2,3}}$ are on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$. We define $g(P_2, K_{2,3}) = g(P_3, K_{2,3}) = \dfrac{1}{4}g(P_2, P_3) = \dfrac{r_2}{4}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{2,3}}$ and $\overline{P_3K_{2,3}}$ is $\dfrac{1}{2}g(P_2, P_3)$.

Now assume that $P_1 - P_2 - P_3$. In this case, we define $g(P_2, P_3) = 4$. In particular, we define $g(P_2, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed. Thus, we define $g(P_2, P_3)$ so that $g(P_2,P_3) > 2s$, where $s = g(P_1,P_2)+g(P_1,K_{1,2})+g(P_2,K_{1,2})$. By Lemma 2, there exists a triangle $\triangle P_2P_3Q_3$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_2P_3Q_3$. Let $K_{2,3}$ be a point in int$(\triangle P_2P_3Q_3)\cap \mathbb{Q}^2$. We define $g(P_2,K_{2,3}) = g(P_3,K_{2,3}) = \frac{1}{4}g(P_2,P_3) = 1$.

We define $g(P_1,P_3)$ by means of segment addition. In particular, $g(P_1,P_3)$= $g(P_1,P_2)+g(P_2,P_3) = 1+4 = 5$. Note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,2}}$, $\overline{K_{1,2}P_2}$, $\overline{P_2K_{2,3}}$, and $\overline{K_{2,3}P_3}$ is $\frac{1}{2}g(P_1,P_3) = \frac{5}{2}$.

Finally, assume that $P_3-P_1-P_2$. This case is similar to the previous case where $P_1-P_2-P_3$. In this case, we define $g(P_1,P_3) = 4$, and $g(P_2,P_3)$= $g(P_1,P_3)+g(P_2,P_1) = 4+1 = 5$. As above, there exists a triangle $\triangle P_1P_3Q_4$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_4$. Let $K_{1,3}$ be a point in int$(\triangle P_1P_3Q_4)\cap\mathbb{Q}^2$. We define $g(P_1,K_{1,3}) = g(P_3,K_{1,3}) = \frac{1}{4}g(P_1,P_3) = 1$. Again, note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{1,2}}$, $\overline{K_{1,2}P_1}$, $\overline{P_1K_{1,3}}$, and $\overline{K_{1,3}P_3}$ is $\frac{1}{2}g(P_2,P_3) = \frac{5}{2}$.

Now assume that P_1, P_2, and P_3 are not collinear. We have several cases, depending on whether $\overleftrightarrow{P_1K_{1,2}}$ or $\overleftrightarrow{P_2K_{1,2}}$ intersect either of the segments $\overline{P_1P_3}$ or $\overline{P_2P_3}$ at any points other than P_1 or P_2.

First assume that neither of the lines $\overleftrightarrow{P_1K_{1,2}}$ nor $\overleftrightarrow{P_2K_{1,2}}$ intersect either of the segments $\overline{P_1P_3}$ or $\overline{P_2P_3}$ at any points other than P_1 or P_2. In this case, we define $g(P_1,P_3) = 4$. Again, we define $g(P_1,P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

By Lemma 2, there exists a triangle $\triangle P_1P_3Q_5$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_5$. Let $K_{1,3}$ be a point in int$(\triangle P_1P_3Q_5)\cap\mathbb{Q}^2$. We define $g(P_1,K_{1,3}) = g(P_3,K_{1,3}) = \frac{1}{4}g(P_1,P_3) = 1$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3}}$ and $\overline{P_3K_{1,3}}$ is $\frac{1}{2}g(P_1,P_3) = 2$.

We define $g(P_2,P_3) = 16$. Again, we define $g(P_2,P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed. Applying Lemma 2, there exists a triangle $\triangle P_2P_3Q_6$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$ intersect the interior of $\triangle P_2P_3Q_6$. Let $K_{2,3}$ be a point in int$(\triangle P_2P_3Q_6)\cap\mathbb{Q}^2$. We define $g(P_2,K_{2,3}) = g(P_3,K_{2,3}) = \frac{1}{4}g(P_2,P_3) = 4$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{2,3}}$ and $\overline{P_3K_{2,3}}$ is $\frac{1}{2}g(P_2,P_3) = 8$.

Next assume that no three of P_1, P_2, P_3, and $K_{1,2}$ are collinear.

Assume that $\overleftrightarrow{P_2K_{1,2}}$ crosses $\overline{P_1P_3}$ at a point $I_{1,3}$, but that $\overleftrightarrow{P_1K_{1,2}}$ does not cross $\overline{P_2P_3}$. In this case, we define $g(P_1, I_{1,3}) = 4$. In particular, we define $g(P_1, I_{1,3})$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

As above, applying Lemma 2, there exists a triangle $\triangle P_1I_{1,3}Q_7$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, and $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1I_{1,3}Q_7$. Let $K_{1,3,1}$ be a point in $\mathrm{int}(\triangle P_1I_{1,3}Q_7)\cap\mathbb{Q}^2$. We define $g(P_1, K_{1,3,1}) = g(I_{1,3}, K_{1,3,1}) = \frac{1}{4}g(P_1, I_{1,3}) = 1$. Note that the arc length in $\mathbb{M}_2$ of the path from P_1 to $I_{1,3}$ along segments $\overline{P_1K_{1,3,1}}$ and $\overline{K_{1,3,1}I_{1,3}}$ is $\dfrac{1}{2}g(P_1, I_{1,3}) = 2$.

We define $g(I_{1,3}, P_3) = 16$. Again, we define $g(I_{1,3}, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

Again, applying Lemma 2, there exists a triangle $\triangle P_3I_{1,3}Q_8$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, and $\overleftrightarrow{I_{1,3}K_{1,3,1}}$ intersect the interior of $\triangle P_3I_{1,3}Q_8$. Let $K_{1,3,2}$ be a point in $\mathrm{int}(\triangle P_3I_{1,3}Q_8) \cap \mathbb{Q}^2$. We define $g(I_{1,3}, K_{1,3,2}) = g(P_3, K_{1,3,2}) = \frac{1}{4}g(P_3, I_{1,3}) = 4$. Note that the arc length in $\mathbb{M}_2$ of the path from P_3 to $I_{1,3}$ along segments $\overline{P_3K_{1,3,2}}$ and $\overline{K_{1,3,2}I_{1,3}}$ is $\dfrac{1}{2}g(P_3, I_{1,3}) = 8$. Also, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along segments $\overline{P_1K_{1,3,1}}$, $\overline{K_{1,3,1}I_{1,3}}$, $\overline{I_{1,3}K_{1,3,2}}$ and $\overline{K_{1,3,2}P_3}$ is $\dfrac{1}{2}g(P_1, P_3)$.

We define $g(P_2, P_3) = 64$. As above, we define $g(P_2, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

Again, applying Lemma 2, there exists a triangle $\triangle P_2P_3Q_9$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, $\overleftrightarrow{I_{1,3}K_{1,3,1}}$, $\overleftrightarrow{P_3K_{1,3,2}}$, and $\overleftrightarrow{I_{1,3}K_{1,3,2}}$ pass through the interior of triangle $\triangle P_2P_3Q_9$. Let $K_{2,3}$ be a point in the interior of triangle $\triangle P_2P_3Q_9$. We define $g(P_2, K_{2,3}) = g(P_3, K_{2,3}) = \frac{1}{4}g(P_2, P_3) = 16$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{2,3}}$ and $\overline{P_3K_{2,3}}$ is $\dfrac{1}{2}g(P_2, P_3) = 32$.

We refer to the segment $\overline{P_1P_2}$ as the *initial segment*. We refer to segments such as $\overline{P_1I_{1,3}}$, $\overline{I_{1,3}P_3}$, and $\overline{P_2P_3}$ as *expansion segments*, since the length of each of these segments is defined by expanding to more than twice the length of all previous segments added together. Similarly, we refer to segments such as $\overline{P_1K_{1,2}}$, $\overline{P_2K_{1,2}}$, and $\overline{P_1K_{1,3}}$ $\overline{P_3K_{1,3}}$ as *contraction segments*, since the length of each of these segment is the contraction of the length of a previous segment.

Assume that $\overleftrightarrow{P_1K_{1,2}}$ crosses $\overline{P_2P_3}$ at a point $I_{2,3}$, but that $\overleftrightarrow{P_2K_{1,2}}$ does not cross $\overline{P_1P_3}$. In this case, we define $g(P_1, P_3) = 4$. As above, we define $g(P_1, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

By applying Lemma 2, there exists a triangle $\triangle P_1P_3Q_{10}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, and $\overleftrightarrow{P_2K_{1,2}}$ pass through the interior of triangle $\triangle P_1P_3Q_{10}$. Let $K_{1,3}$ be a point in the interior of triangle $\triangle P_1P_3Q_{10}$. We define $g(P_1, K_{1,3}) =$

$g(P_3, K_{1,3}) = \frac{1}{4}g(P_1, P_3) = 1$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3}}$ and $\overline{P_3K_{1,3}}$ is $\dfrac{1}{2}g(P_1, P_3) = 2$.

We define $g(P_2, I_{2,3}) = 16$. Again, we define $g(P_2, I_{2,3})$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_2I_{2,3}Q_{11}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1, K_{1,3}}$, and $\overleftrightarrow{P_3K_{1,3}}$ pass through the interior of triangle $\triangle P_2I_{2,3}Q_{11}$. Let $K_{2,3,1}$ be a point in the interior of triangle $\triangle P_2I_{2,3}Q_{11}$. We define $g(P_2, K_{2,3,1}) = g(I_{2,3}, K_{2,3,1}) = \frac{1}{4}g(P_2, I_{2,3}) = 4$. Note that the arc length in $\mathbb{M}_2$ of the path from P_2 to $I_{2,3}$ along segments $\overline{P_2K_{2,3,1}}$ and $\overline{K_{2,3,1}I_{2,3}}$ is $\dfrac{1}{2}g(P_2, I_{2,3}) = 8$.

We define $g(I_{2,3}, P_3) = 64$. As above, we define $g(I_{2,3}, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_3I_{2,3}Q_{12}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1, K_{1,3}}$, $\overleftrightarrow{P_3K_{1,3}}$, $\overleftrightarrow{P_2K_{2,3,1}}$, and $\overleftrightarrow{I_{2,3}K_{2,3,1}}$ pass through the interior of triangle $\triangle P_3I_{2,3}Q_{12}$. Let $K_{2,3,2}$ be a point in the interior of triangle $\triangle P_3I_{2,3}Q_{12}$. We define $g(I_{2,3}, K_{2,3,2}) = g(P_3, K_{2,3,2}) = \frac{1}{4}g(P_3, I_{2,3})$ = 16. Note that the arc length in $\mathbb{M}_2$ of the path from P_3 to $I_{2,3}$ along segments $\overline{P_3K_{2,3,2}}$ and $\overline{K_{2,3,2}I_{2,3}}$ is $\dfrac{1}{2}g(P_3, I_{2,3}) = 32$. Also, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along segments $\overline{P_2K_{2,3,1}}$, $\overline{K_{2,3,1}I_{2,3}}$, $\overline{I_{2,3}K_{2,3,2}}$ and $\overline{K_{2,3,2}P_3}$ is $\dfrac{1}{2}g(P_2, P_3)$.

Assume that $\overleftrightarrow{P_2K_{1,2}}$ crosses $\overline{P_1P_3}$ at a point $I_{1,3}$, and that $\overleftrightarrow{P_1K_{1,2}}$ crosses $\overline{P_2P_3}$ at a point $I_{2,3}$.

In this case, we define we define $g(P_1, I_{1,3}) = 4$. Again, we define $g(P_1, I_{1,3})$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_1I_{1,3}Q_{13}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, and $\overleftrightarrow{P_2K_{1,2}}$ pass through the interior of triangle $\triangle P_1I_{1,3}Q_{13}$. Let $K_{1,3,1}$ be a point in the interior of triangle $\triangle P_1I_{1,3}Q_{13}$. We define $g(P_1, K_{1,3,1}) = g(I_{1,3}, K_{1,3,1}) = \frac{1}{4}g(P_1, I_{1,3}) = 1$.

We define $g(I_{1,3}, P_3) = 16$. Again, we define $g(I_{1,3}, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_3I_{1,3}Q_{14}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, and $\overleftrightarrow{I_{1,3}K_{1,3,1}}$ pass through the interior of triangle $\triangle P_3I_{1,3}Q_{14}$. Let $K_{1,3,2}$ be a point in the interior of triangle $\triangle P_3I_{1,3}Q_{14}$. We define $g(I_{1,3}, K_{1,3,2}) = g(P_3, K_{1,3,2}) = \frac{1}{4}g(P_3, I_{1,3})$ = 4.

We define $g(P_2, I_{2,3}) = 64$. Again, we define $g(P_2, I_{2,3})$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed. There exists a triangle $\triangle P_2I_{2,3}Q_{15}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1, K_{1,3,1}}$, $\overleftrightarrow{I_{1,3}K_{1,3,1}}$, $\overleftrightarrow{I_{1,3}, K_{1,3,2}}$, and $\overleftrightarrow{P_3K_{1,3,2}}$ pass through the interior of triangle $\triangle P_2I_{2,3}Q_{15}$. Let $K_{2,3,1}$ be a point in the interior of triangle $\triangle P_2I_{2,3}Q_{15}$. We define $g(P_2, K_{2,3,1}) = g(I_{2,3}, K_{2,3,1}) = \frac{1}{4}g(P_2, I_{2,3}) = 16$.

We define $g(I_{2,3}, P_3) = 256$. Again, we define $g(I_{2,3}, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_3I_{2,3}Q_{16}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1, K_{1,3,1}}$, $\overleftrightarrow{I_{1,3}K_{1,3,1}}$,$\overleftrightarrow{I_{1,3}, K_{1,3,2}}$, $\overleftrightarrow{P_3K_{1,3,2}}$ $\overleftrightarrow{P_2K_{2,3,1}}$, and $\overleftrightarrow{I_{2,3}K_{2,3,1}}$ pass through the interior of triangle $\triangle P_3I_{2,3}Q_{16}$. Let $K_{2,3,2}$ be a point in the interior of triangle $\triangle P_3I_{2,3}Q_{16}$. We define $g(I_{2,3}, K_{2,3,2}) = g(P_3, K_{2,3,2}) = \frac{1}{4}g(P_3, I_{2,3})$ = 64.

The arc length in $\mathbb{M}_2$ of the path from P_1 to $I_{1,3}$ along segments $\overline{P_1K_{1,3,1}}$ and $\overline{K_{1,3,1}I_{1,3}}$ is $\dfrac{1}{2}g(P_1, I_{1,3})$. The arc length in $\mathbb{M}_2$ of the path from P_3 to $I_{1,3}$ along segments $\overline{P_3K_{1,3,2}}$ and $\overline{K_{1,3,2}I_{1,3}}$ is $\dfrac{1}{2}g(P_3, I_{1,3})$. Also, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along segments $\overline{P_1K_{1,3,1}}$, $\overline{K_{1,3,1}I_{1,3}}$, $\overline{I_{1,3}K_{1,3,2}}$ and $\overline{K_{1,3,2}P_3}$ is $\dfrac{1}{2}g(P_1, P_3)$. The arc length in $\mathbb{M}_2$ of the path from P_2 to $I_{2,3}$ along segments $\overline{P_2K_{2,3,1}}$ and $\overline{K_{2,3,1}I_{2,3}}$ is $\dfrac{1}{2}g(P_2, I_{2,3})$. The arc length in $\mathbb{M}_2$ of the path from P_3 to $I_{2,3}$ along segments $\overline{P_3K_{2,3,2}}$ and $\overline{K_{2,3,2}I_{2,3}}$ is $\dfrac{1}{2}g(P_3, I_{2,3})$. Also, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along segments $\overline{P_2K_{2,3,1}}$, $\overline{K_{2,3,1}I_{2,3}}$, $\overline{I_{2,3}K_{2,3,2}}$ and $\overline{K_{2,3,2}P_3}$ is $\dfrac{1}{2}g(P_2, P_3)$.

Now assume that P_1, P_3, and $K_{1,2}$ are collinear. This case is similar to the case above where P_1, P_2, and P_3 are collinear. Note that in this present case, we are still assuming that P_1, P_2, and P_3 are noncollinear. We have the three possibilities $P_1 - P_3 - K_{1,2}$, $P_1 - K_{1,2} - P_3$, or $P_3 - P_1 - K_{1,2}$.

Assume that $P_1 - P_3 - K_{1,2}$. In this case, since $P_1 - P_3 - K_{1,2}$ in both $\mathbb{M}_2$ and in taxicab geometry, then there exist $r_1, r_2 \in (0, 1) \cap \mathbb{Q}$ such that

(1) $r_1 + r_2 = 1$
(2) $t(P_1, P_3) = r_1t(P_1, K_{1,2})$
(3) $t(P_3, K_{1,2}) = r_2t(P_1, K_{1,2})$

Define $g(P_1, P_3) = r_1g(P_1, K_{1,2})$ and $g(P_3, K_{1,2}) = r_2g(P_1, K_{1,2})$.

There exists a triangle $\triangle P_1P_3Q_{17}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_{17}$. Let $K_{1,3,1}$ be a point in int$(\triangle P_1P_3Q_{17}) \cap \mathbb{Q}^2$. We define $g(P_1, K_{1,3,1}) = g(P_3, K_{1,3,1}) = \dfrac{1}{4}g(P_1, P_3) = \dfrac{r_1g(P_1, K_{1,2})}{4}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3,1}}$ and $\overline{P_3K_{1,3,1}}$ is $\dfrac{1}{2}g(P_1, P_3)$.

Similarly, there exists a triangle $\triangle K_{1,2}P_3Q_{18}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, or $\overleftrightarrow{P_3K_{1,3,1}}$ intersect the interior of $\triangle K_{1,2}P_3Q_{18}$. Let $K_{1,3,2}$ be a point in int$(\triangle K_{1,2}P_3Q_{18}) \cap \mathbb{Q}^2$. We define $g(K_{1,3,2}, K_{1,2}) = g(P_3, K_{1,3,2}) =$

$\frac{1}{4}g(K_{1,2}, P_3) = \frac{r_2 g(P_1, K_{1,2})}{4}$. Thus, the arc length in $\mathbb{M}_2$ of the path from $K_{1,2}$ to P_3 along the segments $\overline{K_{1,2}K_{1,3,2}}$ and $\overline{P_3K_{1,3,2}}$ is $\frac{1}{2}g(K_{1,2}, P_3)$.

Since $P_1 - P_3 - K_{1,2}$, and since P_1, P_2, and $K_{1,2}$ are not collinear, then P_3 is not a point on $\overleftrightarrow{P_2K_{1,2}}$. We define $g(P_2, P_3)$ to be a positive integer such that $g(P_2, P_3) > 2s$, where s is the sum of all previously constructed segments.

Now assume that $P_1 - K_{1,2} - P_3$. In this case, we define $g(K_{1,2}, P_3) = 4$. In particular, we define $g(K_{1,2}, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed. Thus, we define $g(K_{1,2}, P_3)$ so that $g(K_{1,2}, P_3) > 2s$, where $s = g(P_1, P_2) + g(P_1, K_{1,2}) + g(P_2, K_{1,2})$.

We define $g(P_1, P_3)$ by means of segment addition. In particular, $g(P_1, P_3)$= $g(P_1, K_{1,2}) + g(K_{1,2}, P_3)$.

By Lemma 2, there exists a triangle $\triangle P_1K_{1,2}Q_{19}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1K_{1,2}Q_{19}$. Let $K_{1,3,1}$ be a point in int$(\triangle P_1K_{1,2}Q_{19}) \cap \mathbb{Q}^2$. We define $g(K_{1,2}, K_{1,3,1}) = g(P_1, K_{1,3,1}) = \frac{1}{4}g(K_{1,2}, P_1)$.

By Lemma 2, there exists a triangle $\triangle K_{1,2}P_3Q_{20}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, or $\overleftrightarrow{K_{1,2}K_{1,3,1}}$ intersect the interior of $\triangle K_{1,2}P_3Q_{20}$. Let $K_{1,3,2}$ be a point in int$(\triangle K_{1,2}P_3Q_{20}) \cap \mathbb{Q}^2$. We define $g(K_{1,2}, K_{1,3,2}) = g(P_3, K_{1,3,2}) = \frac{1}{4}g(K_{1,2}, P_3)$.

Note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3,1}}$, $\overline{K_{1,2}K_{1,3,1}}$, $\overline{K_{1,3,2}K_{1,2}}$, and $\overline{K_{1,3,2}P_3}$ is $\frac{1}{2}g(P_1, P_3)$.

Again, we have that P_3 is not a point on $\overleftrightarrow{P_2K_{1,2}}$. As above, we define $g(P_2, P_3) > 2s$, where s is the sum of all previously constructed segments.

Finally, assume that $P_3 - P_1 - K_{1,2}$. This case is similar to the previous case where $P_1 - K_{1,2} - P_3$.

In this case, we define $g(P_1, P_3) = 4$. Thus, we define $g(P_1, P_3)$ so that $g(P_1, P_3) > 2s$, where $s = g(P_1, P_2) + g(P_1, K_{1,2}) + g(P_2, K_{1,2})$.

We define $g(P_3, K_{1,2})$ by means of segment addition. In particular, $g(P_3, K_{1,2})$= $g(P_3, P_1) + g(P_1, K_{1,2})$.

As above, there exists a triangle $\triangle P_1P_3Q_{21}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_{21}$. Let $K_{1,3,1}$ be a point in int$(\triangle P_1P_3Q_{21}) \cap \mathbb{Q}^2$. We define $g(P_1, K_{1,3,1}) = g(P_3, K_{1,3,1}) = \frac{1}{4}g(P_1, P_3) = 1$. Again, note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3,1}}$ and $\overline{K_{1,3,1}P_3}$ is $\frac{1}{2}g(P_1, P_3)$.

By Lemma 2, there exists a triangle $\triangle K_{1,2}P_1Q_{22}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, or $\overleftrightarrow{P_3K_{1,3,1}}$ intersect the interior of $\triangle K_{1,2}P_1Q_{22}$. Let $K_{1,3,2}$ be a point in int$(\triangle K_{1,2}P_1Q_{22}) \cap \mathbb{Q}^2$. We define $g(K_{1,2}, K_{1,3,2}) = g(P_1, K_{1,3,2}) = \frac{1}{4}g(K_{1,2}, P_1)$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1

to $K_{1,2}$ along the segments $\overline{P_1K_{1,3,2}}$ and $\overline{K_{1,3,2}K_{1,2}}$ is $\frac{1}{2}g(P_1, K_{1,2})$. Note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_1 to $K_{1,2}$ along the segments $\overline{P_3K_{1,3,1}}$, $\overline{K_{1,3,1}P_1}$, $\overline{P_1K_{1,3,2}}$, and $\overline{K_{1,3,2}K_{1,2}}$ is $\dfrac{1}{2}g(P_3, K_{1,2})$.

Again, we have that P_3 is not a point on $\overleftrightarrow{P_2K_{1,2}}$. As above, we define $g(P_2, P_3) > 2s$, where s is the sum of all previously constructed segments.

The cases where P_2, P_3, and $K_{1,2}$ are collinear are similar to the cases where P_1, P_3, and $K_{1,2}$ are collinear, and are left to the reader.

Assume that there exist n points $P_1, \ldots P_n$ in $\mathbb{Q}^2$ (with $n \geq 3$) such that

(1) For each $i, j \in \{1, \ldots, n\}$, $g(P_i, P_j)$ is defined and is a rational number. More specifically, $g(P_i, P_j)$ is defined by using one of the methods given below.

(2) For each $i, j \in \{1, \ldots, n\}$, with $i \neq j$, if segment $\overline{P_iP_j}$ is not intersected by any previously constructed line, then there exists a point $K_{i,j}$ such that $g(P_i, K_{i,j}) = g(P_j, K_{i,j}) = \dfrac{1}{4}g(P_i, P_j)$, and such that $K_{i,j}$ is not on any other previously constructed line or previously constructed segment. Moreover, no other previously constructed line or previously constructed segment intersects the interior of either segment $\overline{P_iK_{i,j}}$ or $\overline{P_jK_{i,j}}$.

(3) Given $i, j \in \{1, \ldots, n\}$, with $i \neq j$, if there exist points $I_0, I_1, I_2, \ldots, I_{t+1}$ such that $I_0 = P_i$, $I_{t+1} = P_j$, for each $d \in \{1, \ldots, t\}$, I_d is the point of intersection of $\overline{P_iP_j}$ with a previously constructed line (in one of the forms $\overleftrightarrow{P_hP_b}$, $\overleftrightarrow{P_hK_{w,u}}$, or $\overleftrightarrow{P_hK_{w,u,t}}$), for each $d \in \{1, \ldots, t\}$, $I_{d-1} - I_d - I_{d+1}$, and for each $d \in \{0, \ldots, t\}$, no previously constructed line intersects int($\overline{I_dI_{d+1}}$), then for each $d \in \{0, \ldots, t\}$, there exist a point $K_{i,j,d}$ such that $g(I_d, K_{i,j,d}) = g(I_{d+1}, K_{i,j,d}) = \dfrac{1}{4}g(I_dI_{d+1})$, and such that $K_{i,j,d}$ is not on any other previously constructed line or previously constructed segment. Moreover, no other previously constructed line or previously constructed segment intersects the interior of either segment $\overline{I_dK_{i,j,d}}$ or $\overline{I_{d+1}K_{i,j,d}}$.

If P_j is one of the previously constructed points $K_{i,h}$, $K_{i,h,t}$, or I_d, then some (although not necessarily all) of the distances $g(P_i, P_j)$ involving P_i, along with other previously constructed points, will already be defined. However, in general, even if $P_j = K_{i,h}$, $P_j = K_{i,h,t}$, or $P_j = I_d$, then we will still need to define many of the distances $g(P_b, P_j)$ involving P_b, where $P_b \neq P_i$, using the methods given below. For the cases that follow, we assume that P_j is not one of the previously constructed points $K_{i,h}$, $K_{i,h,t}$, or I_d, so that $g(P_i, P_j)$ has not yet been defined. We also assume that $\overline{P_iP_j}$ is not the initial segment $\overline{P_1P_2}$, whose length is already defined to be 1. We now give the methods that must be used to compute $g(P_i, P_j)$. We assume that $i < j$.

(i) Assume that P_i and P_j are not collinear with any other previously constructed points in the form P_k (where $k < j$), $K_{i,h}$, or $K_{i,h,t}$, and that the interior of segment $\overline{P_iP_j}$ is not intersected by any previously constructed line. In this case we define

$g(P_i, P_j)$ to be a positive integer that is strictly greater that twice the sum of the lengths of all previously constructed segments.

(ii) Assume that P_i and P_j are not collinear with any other previously constructed points in the form P_k (where $k < j$), $K_{i,h}$, or $K_{i,h,t}$, but that there exist points $I_0, I_1, I_2, \ldots, I_{t+1}$ such that $I_0 = P_i$, $I_{t+1} = P_j$, for each $d \in \{1, \ldots, t\}$, I_d is the point of intersection of $\overline{P_iP_j}$ with a previously constructed line (in one of the forms $\overleftrightarrow{P_hP_b}$, $\overleftrightarrow{P_hK_{w,u}}$, or $\overleftrightarrow{P_hK_{w,u,t}}$), for each $d \in \{1, \ldots, t\}$, $I_{d-1} - I_d - I_{d+1}$, and for each $d \in \{0, \ldots, t\}$, no previously constructed line intersects int($\overline{I_dI_{d+1}}$). In this case, for each $d \in \{0, \ldots, t\}$, we define $g(I_d, I_{d+1})$ to be a positive integer that is strictly greater that twice the sum of the lengths of all previously constructed segments. We then define $g(P_i, P_j)$ using segment addition. In particular, we define $g(P_i, P_j)$ to be the sum $g(P_i, P_j) = \sum_{d=0}^{t} g(I_d, I_{d+1})$.

(iii) If there exists P_k (where $k < j$) such that $P_i - P_j - P_k$, and there are no other points P_b (where $b < j$) that are between either P_i and P_j or else between P_j and P_k, then $g(P_i, P_j)$ and $g(P_j, P_k)$ are defined using the same constants of proportionality r_1 and r_2 that are used when defining the taxicab distances $t(P_i, P_j)$ and $t(P_j, P_k)$. That is, if $t(P_i, P_j) = r_1 t(P_i, P_k)$ and $t(P_j, P_k) = r_2 t(P_i, P_k)$, then $g(P_i, P_j) = r_1 g(P_i, P_k)$ and $g(P_j, P_k) = r_2 g(P_i, P_k)$. Note that r_1 and r_2 are positive rational numbers.

(iv) We define $g(P_i, P_j)$ and $g(P_j, K_{i,h})$ similarly if there exists a point $K_{i,h}$ such that $P_i - P_j - K_{i,h}$, such that $K_{i,h}$ was constructed previous to P_j, and such that there are no other points P_h (where $h < j$) that are between either P_i and P_j or else between P_j and $K_{i,h}$. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

(v) If there exists P_k (where $k < j$) such that $P_j - P_i - P_k$, and there are no other points P_b (where $b < j$) that are between P_i and P_j, then we define $g(P_i, P_j)$ using one the methods above given in cases (i) through (iv), and we define $g(P_j, P_k)$ by means of segment addition.

(vi) We define $g(P_i, P_j)$ and $g(P_j, K_{i,h})$ similarly if there exists a point $K_{i,h}$ such that $P_j - P_i - K_{i,h}$ and such that $K_{i,h}$ was constructed previous to P_j. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

(vii) We define $g(P_i, P_j)$ and $g(P_j, K_{i,h})$ similarly if there exists a point $K_{i,h}$ such that $P_j - K_{i,h} - P_i$ and such that $K_{i,h}$ was constructed previous to P_j. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

It follows by the way that distance is defined in $\mathbb{M}_2$ together with the way that the points $K_{i,j}$ and $K_{i,j,t}$ are chosen using Lemma 2 that there exists at most one contraction segment on a given line l. The remaining parts of l consist of either the initial segment $\overline{P_1P_2}$ or else expansion segments.

Let P_{n+1} be the next point in the sequence of points (P_j) from $\mathbb{Q}^2$ that comes immediately after the points $P_1, \ldots P_n$. Let $i \in \{1, \ldots, n\}$. We assume that there are no other points P_h (where $h \leq n$) that are between P_i and P_{n+1}. If such points existed, then we would first define $g(P_h, P_{n+1})$, and then define $g(P_i, P_{n+1})$ by means of segment addition. There are several cases when defining the distance

$g(P_i, P_{n+1})$ in $\mathbb{M}_2$. They are similar to the cases given above, but are included here for the sake of completeness.

Case (i): If $P_{n+1} = K_{i,h}$, where $K_{i,h}$ is a previously constructed point used when constructing a contraction segment with P_i, then $g(P_i, P_{n+1})$ has already been defined. In this case, we apply Lemma 2 to get a point $K_{i,n+1}$ such that $g(P_i, K_{i,n+1}) = g(P_{n+1}, K_{i,n+1}) = \dfrac{1}{4} g(P_i, P_{n+1})$.

Assume that $P_{n+1} \neq K_{i,h}$, where $K_{i,h}$ is any of the previously constructed points used when constructing a contraction segment with P_i.

Case(ii): Assume that P_i and P_{n+1} are not collinear with any other previously constructed points in the form P_k (where $k \leq n$), $K_{i,h}$, or $K_{i,h,t}$, and that the interior of segment $\overline{P_i P_{n+1}}$ is not intersected by any previously constructed line. In this case we define $g(P_i, P_{n+1})$ to be a positive integer that is strictly greater that twice the sum of the lengths of all previously constructed segments. As above, we apply Lemma 2 to get a point $K_{i,n+1}$ such that $g(P_i, K_{i,n+1}) = g(P_{n+1}, K_{i,n+1}) = \dfrac{1}{4} g(P_i, P_{n+1})$.

Case (iii): Assume that P_i and P_{n+1} are not collinear with any other previously constructed points in the form P_k (where $k < j$), $K_{i,h}$, or $K_{i,h,t}$, but that there exist points $I_0, I_1, I_2, \ldots, I_{t+1}$ such that $I_0 = P_i$, $I_{t+1} = P_{n+1}$, for each $d \in \{1, \ldots, t\}$, I_d is the point of intersection of $\overline{P_i P_{n+1}}$ with a previously constructed line (in one of the forms $\overleftrightarrow{P_h P_b}$, $\overleftrightarrow{P_h K_{w,u}}$, or $\overleftrightarrow{P_h K_{w,u,t}}$), for each $d \in \{1, \ldots, t\}$, $I_{d-1} - I_d - I_{d+1}$, and for each $d \in \{0, \ldots, t\}$, no previously constructed line intersects int($\overline{I_d I_{d+1}}$). In this case, for each $d \in \{0, \ldots, t\}$, we define $g(I_d, I_{d+1})$ to be a positive integer that is strictly greater that twice the sum of the lengths of all previously constructed segments. We then define $g(P_i, P_{n+1})$ using segment addition. In particular, we define $g(P_i, P_{n+1})$ to be the sum $g(P_i, P_{n+1}) = \displaystyle\sum_{d=0}^{t} g(I_d, I_{d+1})$. For each $d \in \{0, \ldots, t\}$, we apply Lemma 2 to get a point $K_{i,n+1,d}$ such that $g(I_d, K_{i,n+1,d}) = g(K_{i,n+1,d}, I_{d+1}) = \dfrac{1}{4} g(I_d, I_{d+1})$.

Case (iv): If there exists P_k (where $k \leq n$) such that $P_i - P_{n+1} - P_k$, and there are no other points P_b (where $b \leq n$) that are between P_k and P_{n+1}, then $g(P_i, P_{n+1})$ and $g(P_{n+1}, P_k)$ are defined using the same constants of proportionality r_1 and r_2 that are used when defining the taxicab distances $t(P_i, P_{n+1})$ and $t(P_{n+1}, P_k)$. That is, if $t(P_i, P_{n+1}) = r_1 t(P_i, P_k)$ and $t(P_{n+1}, P_k) = r_2 t(P_i, P_k)$, then $g(P_i, P_{n+1}) = r_1 g(P_i, P_k)$ and $g(P_{n+1}, P_k) = r_2 g(P_i, P_k)$. Note that r_1 and r_2 are both positive rational numbers. As above, we apply Lemma 2 to get points $K_{i,n+1,1}$ and $K_{i,n+1,2}$ such that $g(P_i, K_{i,n+1,1}) = g(P_{n+1}, K_{i,n+1,1}) = \dfrac{1}{4} g(P_i, P_{n+1})$ and such that $g(P_k, K_{i,n+1,2}) = g(P_{n+1}, K_{i,n+1,2}) = \dfrac{1}{4} g(P_k, P_{n+1})$.

Case(v): If there exists a point $K_{i,h}$ such that $P_i - P_{n+1} - K_{i,h}$ and such that $K_{i,h}$ was constructed previous to P_{n+1}, and there are no other points P_b (where $b \leq n$) that are between P_{n+1} and $K_{i,h}$, then $g(P_i, P_{n+1})$ and $g(P_{n+1}, K_{i,h})$

are defined using the same constants of proportionality r_1 and r_2 that are used when defining the taxicab distances $t(P_i, P_{n+1})$ and $t(P_{n+1}, K_{i,h})$. That is, if $t(P_i, P_{n+1}) = r_1 t(P_i, K_{i,h})$ and $t(P_{n+1}, K_{i,h}) = r_2 t(P_i, K_{i,h})$, then $g(P_i, P_{n+1}) = r_1 g(P_i, K_{i,h})$ and $g(P_{n+1}, K_{i,h}) = r_2 g(P_i, K_{i,h})$. Again, both r_1 and r_2 are positive rational numbers. Again, we apply Lemma 2 to get points $K_{i,n+1,1}$ and $K_{i,n+1,2}$ such that $g(P_i, K_{i,n+1,1}) = g(P_{n+1}, K_{i,n+1,1}) = \frac{1}{4} g(P_i, P_{n+1})$ and such that $g(K_{i,h}, K_{i,n+1,2}) = g(P_{n+1}, K_{i,n+1,2}) = \frac{1}{4} g(K_{i,h}, P_{n+1})$. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

Case(vi): If there exists P_k (where $k \leq n$) such that $P_{n+1} - P_i - P_k$, then we define $g(P_i, P_{n+1})$ using one the methods above given in cases (ii), (iii), (iv), or (v), and we define $g(P_{n+1}, P_k)$ by means of segment addition. As above, we apply Lemma 2 to get a point $K_{i,n+1}$ such that $g(P_i, K_{i,n+1}) = g(P_{n+1}, K_{i,n+1}) = \frac{1}{4} g(P_i, P_{n+1})$. Note that a point $K_{i,k}$ such that $g(P_i, K_{i,k}) = g(P_k, K_{i,k}) = \frac{1}{4} g(P_i, P_k)$ has already been previously constructed.

Case(vii): If there exists a point $K_{i,h}$ such that $P_{n+1} - P_i - K_{i,h}$ and such that $K_{i,h}$ was constructed previous to P_{n+1}, then we define $g(P_i, P_{n+1})$ and $g(P_{n+1}, K_{i,h})$ similar to case (vi). Again, we apply Lemma 2 to get points $K_{i,n+1,1}$ and $K_{i,n+1,2}$ such that $g(P_i, K_{i,n+1,1}) = g(P_{n+1}, K_{i,n+1,1}) = \frac{1}{4} g(P_i, P_{n+1})$ and such that $g(K_{i,h}, K_{i,n+1,2}) = g(P_i, K_{i,n+1,2}) = \frac{1}{4} g(K_{i,h}, P_i)$. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

Case(viii): If there exists a point $K_{i,h}$ such that $P_{n+1} - K_{i,h} - P_i$ and such that $K_{i,h}$ was constructed previous to P_{n+1}, then we again define $g(P_i, P_{n+1})$ and $g(P_{n+1}, K_{i,h})$ similar to case (vi). In this case the roles of P_i and $K_{i,h}$ are reversed from Case (vii). We apply Lemma 2 to get points $K_{i,n+1,1}$ and $K_{i,n+1,2}$ such that $g(K_{i,h}, K_{i,n+1,1}) = g(P_{n+1}, K_{i,n+1,1}) = \frac{1}{4} g(K_{i,h}, P_{n+1})$ and such that $g(K_{i,h}, K_{i,n+1,2}) = g(P_i, K_{i,n+1,2}) = \frac{1}{4} g(K_{i,h}, P_i)$. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

In any of the above cases, we see that $g(P_i, P_{n+1})$, and where appropriate $g(P_{n+1}, K_{i,h})$ or $g(P_{n+1}, K_{i,h,t})$, are defined. In this way we can recursively define distance between any two points in the model $\mathbb{M}_2$. We also see that between any two points P_i and P_j in $\mathbb{M}_2$, there exists a path whose arc length in $\mathbb{M}_2$ is $\frac{1}{2} g(P_i, P_j)$. By repeatedly applying Lemma 2, we can construct paths from P_i and P_j whose arc lengths in $\mathbb{M}_2$ are arbitrarily small. Thus, there is no shortest path in $\mathbb{M}_2$ from P_i to P_j, and consequently, $\mathbb{M}_2$ is nowhere geodesic.

We next show that the congruence axioms for line segments hold in $\mathbb{M}_2$.

It follows immediately by the way that distance $g(P_i, P_j)$ (and consequently congruence) is defined in $\mathbb{M}_2$ that congruence axioms (ii) and (iii) for line segments hold in $\mathbb{M}_2$.

We now prove congruence axiom (1) for line segments. Let $A, B, C \in \mathbb{Q}^2$, and let $r = \overrightarrow{CT}$ denote a ray in $\mathbb{M}_2$ originating at C. We will prove that there exists a unique point D on r such that $g(A, B) = g(C, D)$. If $A = B$, then we let $C = D$. Assume that $A \neq B$, and therefore that $g(A, B)$ is a strictly positive rational number.

We now construct a subsequence of points (P_{j_t}) from (P_j) that are on r. We define the interior of r to be all points on r other than the point C. Let $C = P_{j_0}$. Let P_{j_1} denote the first point from the sequence (P_j) that is in the interior of r. Let P_{j_2} denote the first point from the sequence (P_j) such that $C - P_{j_1} - P_{j_2}$. Note by our choices of the points P_{j_1} and P_{j_2} that it must be the case that P_{j_2} comes after P_{j_1} in the sequence (P_j). Assume that we have chosen points P_{j_0}, P_{j_1}, P_{j_2}, …, P_{j_t} from (P_j) such that

(1) For each $i = 0, \ldots, t-1$, we have that P_{j_i} comes before $P_{j_{i+1}}$ in the sequence (P_j).
(2) For each $i = 0, \ldots, t-2$, we have $P_{j_i} - P_{j_{i+1}} - P_{j_{i+2}}$.

In particular, for each $i = 1, \ldots, t-1$, $P_{j_{i+1}}$ is the first point from the sequence (P_j) such that $C - P_{j_i} - P_{j_{i+1}}$.

Let $P_{j_{t+1}}$ denote the first point from the sequence (P_j) such that $C - P_{j_t} - P_{j_{t+1}}$. Again, it must be the case that P_{j_t} comes before $P_{j_{t+1}}$ in the sequence (P_j).

Thus, we have a subsequence (P_{j_t}) of (P_j) such that

(1) For each $i \geq 1$, we have that P_{j_i} comes before $P_{j_{i+1}}$ in the sequence (P_j).
(2) For each i, we have $P_{j_i} - P_{j_{i+1}} - P_{j_{i+2}}$.

In particular, for each $i \geq 1$, $P_{j_{i+1}}$ is the first point from the sequence (P_j) such that $C - P_{j_i} - P_{j_{i+1}}$.

Let P_{j_b} denote the first point from the subsequence (P_{j_t}) such that $g(C, P_{j_b}) \geq g(A, B)$. Since there exists at most one contraction segment on line $\overleftrightarrow{CT}$, then it follows by the way that the lengths of expansion segments are constructed that such a point P_{j_b} exists. If $g(C, P_{j_b}) = g(A, B)$, then we let $D = P_{j_b}$. Assume that $g(C, P_{j_b}) > g(A, B)$. If $b = 1$, then let $v = 0$. Assume that $b \geq 2$. In this case, we let $v = \displaystyle\sum_{n=1}^{b-1} g(P_{j_{n-1}}, P_{j_n}) < g(A, B)$. Let $u = \dfrac{g(A, B) - v}{g(P_{j_{b-1}}, P_{j_b})} \in \mathbb{Q} \cap (0, 1)$.

We leave it to the reader to check, using density of $\mathbb{Q}^2$ in $\mathbb{R}^2$, and therefore on the line $\overleftrightarrow{CT}$, that there exists a unique point D such that $D \in \mathbb{Q}^2$, such that D is between $P_{j_{b-1}}$ and P_{j_b}, and such that $t(P_{j_{b-1}}, D) = u(t(P_{j_{b-1}}, P_{j_b}))$.

It follow by the way that the subsequence (P_{j_t}) was constructed that D comes after P_{j_b} in the sequence (P_j). It now follows immediately by the way that distance is defined in $\mathbb{M}_2$ that D is the unique point in the interior of segment $\overline{P_{j_{b-1}} P_{j_b}}$ such that $g(P_{j_{b-1}}, D) = u(g(P_{j_{b-1}}, P_{j_b})) = g(A, B) - v$.

Moreover, it now follows immediately by the way that distance is defined in $\mathbb{M}_2$ (via segment addition) that D is the unique point on ray r such that $g(C, D) = v + u(g(P_{j_{b-1}}, P_{j_b})) = v + (g(A, B) - v) = g(A, B)$.

Thus, it follows that congruence axiom (1) for line segments holds in $\mathbb{M}_2$.

Lemma 5. *Given a triangle $\triangle P_iP_jP_t$, then exactly one of the following is true:*

(1) *At least one of the sides $\overline{P_iP_j}$ is an expansion segment whose length is defined after the lengths of the other two sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are defined.*
(2) *At least one of the sides $\overline{P_iP_j}$ contains as a subsegment an expansion segment whose length is defined after the lengths of the other two sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are defined.*
(3) *The segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are contraction segments such that the arc length in $\mathbb{M}_2$ along the segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ is $\dfrac{1}{2}g(P_i, P_j)$.*

Proof. By the way that the points $K_{i,j}$ and $K_{i,j,t}$ are chosen using Lemma 2, it can not be the case that all three sides of a triangle $\triangle P_iP_jP_t$ are subsegments of contraction segments. Suppose that $\overline{P_iP_j}$ is a subsegment of a contraction segment $\overline{P_{z_1}K_1}$, $\overline{P_iP_t}$ is a subsegment of a contraction segment $\overline{P_{z_2}K_2}$, and $\overline{P_jP_t}$ is a subsegment of a contraction segment $\overline{P_{z_3}K_3}$. For this to happen, P_i is the point of intersection of $\overline{P_{z_1}K_1}$ and $\overline{P_{z_2}K_2}$, and P_t is the point of intersection of $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$. We may assume that K_3 is chosen and consequently $\overline{P_{z_3}K_3}$ is constructed using Lemma 2 after both of K_1 and K_2 are chosen and $\overline{P_{z_1}K_1}$ and $\overline{P_{z_2}K_2}$ are constructed. However, when choosing K_3 to construct the contraction segment $\overline{P_{z_3}K_3}$, we apply Lemma 2 and choose K_3 so that no line containing a previously constructed contraction segment intersects $\overline{P_{z_3}K_3}$. This, contradicts the above statement that P_i is the point of intersection of $\overline{P_{z_1}K_1}$ and $\overline{P_{z_2}K_2}$, and P_t is the point of intersection of $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$.

A similar argument applies if segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are subsegments of contraction segments $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$, respectively, and the length of $\overline{P_iP_j}$ is defined before $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$ are constructed. In this case K_2 and K_3 are chosen using Lemma 2 so that no previously constructed line such as $\overleftrightarrow{P_iP_j}$ intersects either of $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$. However, in this case we have that P_i is the point of intersection of $\overleftrightarrow{P_iP_j}$ and $\overline{P_{z_2}K_2}$, and P_j is the point of intersection of $\overleftrightarrow{P_iP_j}$ and $\overline{P_{z_3}K_3}$, respectively, a contradiction. Thus, if segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are subsegments of contraction segments $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$, respectively, then $\overline{P_iP_j}$ must contain, as a subsegment, an expansion segment $\overline{P_uP_v}$ whose length is defined after $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$ have been constructed. More generally, the length of $\overline{P_uP_v}$ is defined after the lengths of the other two sides of $\triangle P_iP_jP_t$ have been defined as proportions of the length of the contraction segments $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$.

Moreover, a similar argument applies if the segment $\overline{P_jP_t}$ is a subsegment of a contraction segment $\overline{P_{z_3}K_3}$, respectively, and the lengths of $\overline{P_iP_j}$ and $\overline{P_iP_t}$ are defined before $\overline{P_{z_3}K_3}$ is constructed. In this case K_3 is chosen using Lemma 2 so that no previously constructed lines such as $\overleftrightarrow{P_iP_j}$ or $\overleftrightarrow{P_iP_t}$ intersect $\overline{P_{z_3}K_3}$. However, in this case we have that P_j is the point of intersection of $\overleftrightarrow{P_iP_j}$ and $\overline{P_{z_3}K_3}$, and P_t is the point of intersection of $\overleftrightarrow{P_iP_t}$ and $\overline{P_{z_3}K_3}$, respectively, a contradiction. Thus, if segment $\overline{P_jP_t}$ is a subsegment of a contraction segment $\overline{P_{z_3}K_3}$, then one of $\overline{P_iP_j}$ or $\overline{P_iP_t}$ must contain, as a subsegment, an expansion segment $\overline{P_uP_v}$ whose length is defined after $\overline{P_{z_3}K_3}$ has been constructed. More generally, the length of $\overline{P_uP_v}$ is defined after the lengths of the other two sides of $\triangle P_iP_jP_t$ have been defined.

If $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are themselves contraction segments, then by the way that contraction segments are defined, we have that the arc length in $\mathbb{M}_2$ along the segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ is $\dfrac{1}{2}g(P_i, P_j)$. Note that this is the only case where $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are contraction segments and the length of $\overline{P_iP_j}$ is defined before $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are constructed.

Next assume that there exist points K_4, K_5, and K_6 in the interiors of sides $\overline{P_iP_j}$, $\overline{P_iP_t}$, and $\overline{P_jP_t}$, respectively, such that each of the segments $\overline{P_iK_4}$, $\overline{P_iK_5}$, and $\overline{P_jK_6}$ are either themselves contraction segments or else contain a contraction segment as a subsegment. In this case, each of $\overline{P_iP_j}$, $\overline{P_iP_t}$, and $\overline{P_jP_t}$ contain as a subsegment at least one expansion segment. By the way that contraction segments are constructed using Lemma 2, it must be the case that these expansion segments are constructed after the contraction segments are constructed. Thus, in this case one of the sides contains as a subsegment an expansion segment whose length is defined after the lengths of the other two sides are defined.

A similar result occurs if there exist two points K_5 and K_6 in the interiors of sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$, respectively, such that each of the segments $\overline{P_iK_5}$ and $\overline{P_jK_6}$ are either themselves contraction segments or else contain a contraction segment as a subsegment, and the length of $\overline{P_iP_j}$ is defined before the contraction segments are constructed. In this case one of the sides $\overline{P_iP_t}$ or $\overline{P_jP_t}$ contains as a subsegment an expansion segment whose length is defined after the lengths of the other two sides are defined.

Finally, if there exists a point K_6 in the interior of side $\overline{P_jP_t}$, respectively, such that the segment $\overline{P_jK_6}$ is either itself a contraction segment or else contains a contraction segment as a subsegment, and the lengths of $\overline{P_iP_t}$ and $\overline{P_iP_j}$ are defined before the contraction segment is constructed. In this case, the side $\overline{P_jP_t}$ contains as a subsegment an expansion segment whose length is defined after the lengths of the other two sides are defined.

The only remaining possibility is that one of the sides $\overline{P_iP_j}$, $\overline{P_iP_t}$, and $\overline{P_jP_t}$ is an expansion segment whose length is defined after the lengths of the other two sides are defined. $\square$

Assume that we are given triangle $\triangle P_iP_jP_t$. Assume that one of the sides $\overline{P_iP_j}$ is an expansion segment whose length is defined after the lengths of the other two sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are defined. In this case, $g(P_i, P_j)$ is strictly larger than twice the sum of all segments whose lengths have been defined previously. More precisely, $g(P_i, P_j) > 2s$, where s denotes the sum of the lengths of all segments whose lengths have been defined previously. Since $g(P_i, P_t)$ and $g(P_t, P_j)$ are comprised of segments whose lengths are defined prior to defining the length of $\overline{P_iP_j}$, then $s \geq g(P_i, P_t) + g(P_t, P_j)$. Thus, $g(P_i, P_j) > 2s \geq g(P_i, P_t) + g(P_t, P_j)$.

Next assume that at least one of the sides $\overline{P_iP_j}$ contains a subsegment $\overline{P_uP_v}$ such that $\overline{P_uP_v}$ is an expansion segment whose length is defined after the lengths of the other two sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are defined. Similar to the previous case above, we have that $g(P_u, P_v) > 2s$, where s denotes the sum of the lengths of all segments whose lengths have been defined previously. Again, since $g(P_i, P_t)$ and $g(P_t, P_j)$ are comprised of segments whose lengths are defined prior to defining the length of $\overline{P_uP_v}$, then $s \geq g(P_i, P_t) + g(P_t, P_j)$. Thus, $g(P_i, P_j) \geq g(P_u, P_v) > 2s \geq g(P_i, P_t) + g(P_t, P_j)$.

Next assume that $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are contraction segments such that the arc length in $\mathbb{M}_2$ along the segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ is $\dfrac{1}{2}g(P_i, P_j)$. In this case, $g(P_i, P_j) > \frac{1}{2}g(P_i, P_j) = g(P_i, P_t) + g(P_t, P_j)$.

Thus, in any of these cases, the triangle inequality fails for $\triangle P_iP_jP_t$ in $\mathbb{M}_2$.

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