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Is The Mystery of Morley's Trisector Theorem Resolved?

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Dedicated to all those mathematicians who have been enthusiastic about the above theorem and have published a proof to the theorem. In particular, to John Conway, whose proof prevailed upon me to return to this theorem after more than half a century. I should also remember Brian Stonebridge, who has published a few different proofs of Morley's Theorem and sadly has recently passed away. He was so enthusiastic about the theorem that even while confined to bed with hip fractures, he presented a new proof to the theorem and shortly before his death, dispatched it to me.

Abstract. In this paper we discuss with some reasons why the above possible mystery in the title is resolved. In particular, we prove Morley's Theorem is, in fact, a natural consequence of an overlooked simple result concerning a general property of angle bisectors as loci. We try to show with emphasis why one cannot expect a simpler proof of Morley's Theorem than a particular given proof of the theorem in the literature. Finally, in view of the above overlooked fact, we observe that two known problems in the literature can naturally be generalized and turned into two corollaries with no needed proofs. More applications of this overlooked fact is also observed.

Let us first recall that if the adjacent trisectors of a triangle $\triangle ABC$ intersect at points X, Y, and Z as in Figure (a), then the triangle $\triangle XYZ$ is called the Morley triangle of triangle $\triangle ABC$. Frank Morley, an English mathematician, while studying some properties of cardioid in [15], has given the following incredible theorem (note, the theorem is proved by considering the locus of the centre of a cardioid which touches the sides of a given triangle).

Morley's Theorem (1899). In every triangle the Morley triangle is equilateral.

Some authors believe this theorem is one of the most surprising and mysterious twentieth century results in Euclidean geometry, see for example [2], [7], and [21]. Based on the criteria: the place of the theorem in the literature, the quality of the proofs, the unexpectedness of the result, Morley's Theorem is in a list of "The

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Hundred Greatest Theorems in Mathematics". [1] Although, Morley in [15] did not give an elementary proof to his theorem, but soon after only a decade an elementary proof appeared in [16]. Since then many other elementary proofs of Morley's Theorem have been published, [2] where there are 27 different proofs of the theorem (note, still there are many more). Béla Bollobás, Alain Connes, J, John Conway, Edsger Dijksta, D.J. Newmann, Roger Penrose, M.R.F. Smyth, Brian Stonebridge (just to mention a few) are among the authors who have given some of the above 27 different proofs. These proofs are mostly trigonometric, analytic, algebraic, or contain some unexplained elements in their reasoning, and those which are backwards, usually start with an equilateral triangle $\triangle DEF$ and construct a triangle $\triangle ABC$ with arbitrary angles 3α, 3β and 3γ, where $\alpha + \beta + \gamma = 60°$, such that triangle $\triangle DEF$ is its Morley triangle. Thus the triangle $\triangle ABC$ would be similar to a given triangle and the proof is finished, see for example [19],[6],[10],[17] and one of the two proofs of Bollobás, among the above 27 proofs, for five such proofs. In some of these backwards proofs, the authors usually construct three triangles around the equilateral triangle $\triangle DEF$ with some specified angles without any explanations for the specification of these angles. Of course these proofs are just fine, but unfortunately with lack of motivation and for this reason, some authors rightly criticize these kind of proofs, see for example [3] and [13, P 74]. Some of the authors who use trigonometry, give a proof to the theorem, by finding a nice symmetric formula for the length of a side of Morley triangle of a triangle $\triangle ABC$ (note, this length equals to $8Rsin\frac{A}{3}sin\frac{B}{3}sin\frac{C}{3}$, where R denotes the circumradius of the triangle $\triangle ABC$, see [24]). The reader should be reminded that, using elementary Cartesian analysis, this length was also obtained by C.N. Mills. His complete proof occupied some twenty sheets of papers, see [24, The Editor Note]. John Conway in [6] claims that "of all the many proofs of this theorem his is indisputably the simplest". By modifying this proof in [12], it is shown that this modified proof (still, called Conway's proof), which also avoids using similarity, is simpler than Conway's, and therefore it was, comparatively, the simplest proof of Morley's Theorem, for the time being, at that time. In [12], it is also admitted that one does not know what happens tomorrow, with regard to the simplicity of the proof, see the last comment in [12]. Although, in [8] the authors already claim, in the title of the article, that Morley's Theorem is no longer mysterious, but we would like to offer here some detailed reasoning for this true claim (at least, to us). In particular, our main aim is to observe and claim with emphasis that the mystery of Morley's Theorem, is indeed, resolved. To this end, by just looking more carefully, at the simple proposition in [12], which is called Bisector Proposition in [8], we may infer that Morley's Theorem is nothing but a straightforward corollary of this simple proposition. Surprisingly, it seems this proposition, which is a natural generalization of the classical property of the bisectors (i.e., a point lies on the bisector of an angle if and only if it is equidistant from the sides of the angle), is

[1] See `http://pirate.shu.edu/ kahlnath/Top100.html`.

[2] See, e.g., `http:// www.cut-the-knot.org/triangle/Morley/sb.shml`, [2-4, 7-14]

overlooked in all these years, before its appearance in [12]. Let us cite a comment which is made in [13, Last line, P 73]: with all the various proofs in the literature, still a geometric, concise and logically transparent proof of Morley's Theorem is desirable. A similar comment is also given in [21, L 3, P 1]. In what follows, we aim to emphasize on the effectiveness of, Bisector Proposition as a tool, and on its unique role in connection with Morley's Theorem. We also believe with some reasoning that this demanded proof in [13], [21], which in our opinion, could have been the modified proof in [12], at that time, is already in [8], with some improvement. We should also emphasize that this proof is geometric, concise, transparant, and simpler than the proof in [12], which in turn, as shown there, is simpler than Conway's, where to many authors including Conway himself, Conway's was the simplest possible until 2014, see [6]. Although, by reading [12] and [8] carefully, this claim needs no proof, but one feels this important fact (at least to us) should be better recorded. Since we are trying to give our reasoning with scrupulous care, the reader might notice some unnecessary details in our arguments. We should recall here that each vertex of the Morley triangle of a given triangle, which is the intersection point of a pair of trisectors, lies trivially on the bisector of the angle formed by the intersection of two other trisectors (e.g., Z lies on the bisector of the angle formed by the intersection of trisectors BX and AY, see Figure (a)). Consequently, each vertex of the Morley triangle takes the position of point A in the next proposition. Hence, the conditions (1), (3), in the following proposition, immediately give us a natural criterion , as in the corollary which follows the proposition, for the Morley triangle of any triangle, to be equilateral. In fact, what Morley's Theorem asserts is the fact that this natural criterion holds in every triangle.

The next proposition, says the point A lies on the bisector of the angle $\angle xOy$ if, and only if, any one of the conditions (2) or (3) in the proposition implies the other one. We notice, in the classical fact, the condition (2) is stated, in the particular case, when B, C are the feet of perpendiculars from the point A to the arms of angle $\angle xOy$, and the condition (3) is stated, in the particular case, that the angles $\angle OBA$, $\angle OCA$, are both right angle, in which case, they are both equal and supplementary angles (i.e., condition (3) below is naturally satisfied in the particular case).

Bisector Proposition. Suppose that A is a point inside the angle $\angle xOy$ and B, C are two points on the arms Ox and Oy, respectively. Then if any two of the following hold, so does the third.

(1) A lies on the bisector of the angle $\angle xOy$.
(2) $AB = AC$.
(3) Angles $\angle OBA$ and $\angle OCA$ are either equal or supplementary angles.

We should remind the reader that, it is manifest that, the angles $\angle OBA$ and $\angle OCA$ satisfy condition (3) in the above proposition if, and only if, their adjacent angles satisfy this condition, too. This evident observation is in fact, what is used when Bisector Proposition is invoked in [12], to deal with the proof of the corollary which follows. This corollary which is given right after the proof of the above proposition in [12], is not stated there as a corollary. Indeed, it is given there informally to refute a claim by Cain in [3], that specification of the angles in Conway's proof is not justified. We should remind the reader that the following corollary which gives "a necessary and sufficient condition" for the Morley triangle of any triangle to be equilateral, leaves no unexplained steps whatsoever (unlike, the other backwards proofs) in the proof of Morley's Theorem in [8].

The next corollary, which relies only on the above proposition, is as good as Morley's Theorem itself, for it is the only result in Euclidean geometry, so far, which gives a natural "necessary and sufficient condition", for the Morley triangle of any triangle to be equilateral, before knowing the validity of Morley's Theorem.

Corollary 1. In the triangle $\triangle ABC$, in Figure (a), let $\angle A = 3\alpha$, $\angle B = 3\beta$, and $\angle C = 3\gamma$. If the intersection of the adjacent trisectors of the angles of this triangle are X, Y, and Z as in Figure (a). Then the triangle $\triangle XYZ$ (i.e., the Morley triangle of triangle $\triangle ABC$) is equilateral if, and only if, the base angles of the triangles $\triangle AZY$, $\triangle BXZ$, and $\triangle CXY$ consist of three equal pairs (i.e., $\angle 1 = \angle 4 = 60^\circ + \gamma$, $\angle 2 = \angle 5 = 60^\circ + \beta$, and $\angle 3 = \angle 6 = 60^\circ + \alpha$), see Figure (a).

The reader must be reminded that the values of the above pairs of angles are determined, with a purely geometric method (thanks to Bisector Proposition), for the first time in the history of Morley's Trisector Theorem, in [12]. In any configuration related to Morley's Theorem we have seven smaller triangles inside the original one. Why should we only be asking for the values of angles of the Morley's triangle?, see also Newmann's comments in his proof among the above 27 proofs. In this regards, let us briefly recall the role of Bisector Proposition in determining the values of the angles of all these seven triangles. Let us assume the Morley triangle, of a triangle , $\triangle ABC$ say, to be equilateral and try to find the values of angles $\angle 1$ up to $\angle 6$, in a relevant configuration, as in Figure (a). We may easily form a system of 6 linear equations in terms of these values. For the sake of completeness we may write down these equations: $\angle 1 + \angle 2 = 180^\circ - \alpha$, $\angle 2 + \angle 3 = 180^\circ - \gamma$, $\angle 3 + \angle 4 = 180^\circ - \beta$, $\angle 4 + \angle 5 = 180^\circ - \alpha$, $\angle 5 + \angle 6 = 180^\circ - \gamma$, $\angle 6 + \angle 1 = 180^\circ - \beta$. Manifestly, these equations are not independent. Indeed, any one of these equations can be derived from the other five, and therefore any one of the equations can be removed without affecting the solution of the system. Consequently, we have a system of five linear equations with six unknown values of angles, whose unique solution cannot usually be determined, by ordinary elimination method. Although, there is no unanimity of opinion, on any reason, for the mysteriousness of Morley's Theorem, perhaps the latter observation seems to be a good possible reason as to why this theorem is called mysterious, by some authors. It should be emphasized that Bisector Proposition plays an effective and indispensable role in showing that the above system has a unique solution, and conversely due to Bisector Proposition again, this solution without any further calculation whatsoever for the angles of the Morley's triangle, immediately shows that the Morley's triangle is equilateral, see the proof of the above corollary, in [12].

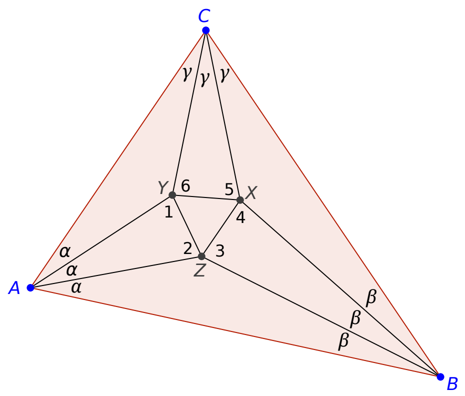

Figure (a)

Corollary 2 (Morley's Theorem). The Morley triangle of every triangle is equilateral.

If we follow the proof in [8], we notice that Bisector Proposition, which is responsible for the proof of Corollary 1, is also responsible for every single step in this proof, without using any extra mathematics whatsoever. In fact, Bisector Proposition shows in [8], one can simultaneously construct any triangle with its trisectors in a few steps, in a motivated simple way, without drawing any extra single line, to automatically, obtaining an equilateral triangle as its Morley triangle (note, in this proof unlike other backwards proofs, one does not start with an equilateral triangle). In fact, it may be considered as drawing the trisectors in the triangle, in a motivated way, to get an equilateral triangle, directly, as the Morley triangle of the triangle (once again, it should be emphasized that in this drawing not a single extra line is drawn).

Envoi: some authors, use the words mystery, miraculous, magic and some other synonyms to these words, in connection with Morley's Theorem and are wondering why the ancient Greeks did not discover this theorem. They generally blame the concept of trisectors for this failure. They believe the ancient Greeks, apart from having problems with the existence of trisectors (i.e., trisecting an angle, in general, with just a compass and an unmarked straightedge), expected to know some properties of trisectors similarly to those of bisectors. But it seems, if only the ancient Greeks noticed that, each vertex of the Morley triangle lies clearly on the bisector of the angle formed by a pair of trisectors, whose intersection is not that vertex, they might have been motivated to search for new properties of bisectors.

Moreover, if they could have also guessed the Bisector Proposition, which genuinely and naturally belongs to their era, then they could have easily discovered Morley's Theorem. Although at the same time, one should admit that, it seems the ancient Greeks have not ever dealt with any result in geometry, whose related possible configuration, is not constructible with Euclidean instruments. However, I do believe the fact that Bisector Proposition (a natural elementary geometric fact, which also seems to be a key factor related to Morley' Theorem) is gone unnoticed in all these years, is more mysterious than the missing of Morley's Theorem itself, by the ancient Greeks! Therefore, we should admit, it seems as though, all these years we were waiting for the Bisector Proposition to appear, to give us a natural, transparent geometric proof of Morley's Theorem. Incidentally, this proof, see [8], needs only the simplest possible configuration (i.e., a figure consisting merely of the triangle itself and its trisectors). Moreover, it also uses only the most elementary tools (i.e., SAS, ASA and RHS). Shouldn't we ask that: Can a proof of this theorem be expected, in the future, which avoids less elementary tools? Shouldn't we admit that Bisector Proposition has indeed resolved the mystery of Morley's Theorem? To re-emphasizing on the effectiveness of Bisector Proposition as a useful tool, in general, not only with regards to Morley's Theorem, I could not help recalling the following well known old problems to show how, by invoking Bisector Proposition, the two problems can be generalized and turned into two natural evident corollaries of this proposition. Some more applications of the proposition are also observed.

Problem 1 ([20, Problems 9, 18, P 236, 267], [1, Problem 24]): Let P be the center of the square constructed on the hypotenuse AC of the right triangle $\triangle ABC$. Prove that BP bisects angle $\angle ABC$.

Corollary 3 . Let P be the center of the square constructed on the side AC of the triangle $\triangle ABC$. Then BP bisects the angle $\angle ABC$ if, and only if, the triangle $\triangle ABC$ is either a right-angled triangle, with $\angle B$ as the right angle, or an isosceles one with the apex B.

Proof. It is really evident by Bisector Proposition (just note, $PA = PC$).

Terence Tao in [22, pp. 52-55], has given a trigonometric solution with almost three pages of justifications for his solution to the next problem. But, in view of Bisector Proposition, we observe briefly, this problem does not really need any solution at all. Naturally, Bisector Proposition was not available to Tao , at that time, because the proposition was already overlooked then. Similarly to Problem 1, this problem, can also be generalized and stated as an immediate corollary of Bisector Proposition. However, in what follows, we present a proof for this corollary with some unnecessary details, just to repeat the effectiveness of Bisector Proposition as a useful tool.

Problem 2 ([22, Problem 4.2, p. 52]): In a triangle $\triangle BAC$ the bisector of the angle at B meets AC at D; the angle bisector of C meets AB at E. These bisectors meet at O. Suppose that $OD = OE$. Prove that either $\angle A = 60°$ or that the triangle $\triangle BAC$ is isosceles (or both).

Corollary 4. In a triangle $\triangle ABC$ let the bisectors of angles $\angle B$, $\angle C$, intersect the opposite sides at D and E, respectively, and take O to be the intersection of these bisectors. Then $OD = OE$ if, and only if, either the triangle $\triangle BAC$ is isosceles with apex A or $\angle A = 60°$.

Proof. Just draw any configuration which satisfies our assumption, then the proof may go as follows : Let us first assume that, $OD = OE$. Since the point O lies on the bisector of $\angle A$, then by part (3), of Bisector Proposition, we infer that either $\angle AEO = \angle ADO$, in which case, it implies $\angle\frac{C}{2} + \angle B = \angle\frac{B}{2} + \angle C$, i.e., the triangle $\triangle BAC$ is isosceles with apex A, or $\angle ADO$ and $\angle AEO$ are supplementary angles, in which case, it implies $\angle A$ and $\angle DOE$ must also be supplementary angles. Hence $\angle A = 60°$ (note, $\angle DOE = \angle COB = 90° + \angle\frac{A}{2}$). Conversely, let the triangle $\triangle BAC$ be isosceles with $\angle B = \angle C$, then clearly $OD = BD - BO = CE - CO = OE$. Finally, if $\angle A = 60°$, then $\angle A + \angle DOE = 180°$, for $\angle DOE = \angle COB = 90° + \frac{A}{2} = 120°$. Consequently, the angles $\angle ADO$ and $\angle AEO$ must also be supplementary angles. Since O lies on the bisector of angle $\angle A$, then in view of, parts (1), (3), of Bisector Proposition, we immediately infer that $OD = OE$, and we are done.

Let us assume that the above triangle $\triangle ABC$ is non-isosceles and $\angle A \neq 60°$. Then one may digress for a moment and ask a natural question of what happens to the comparability of OD and OE (i.e., considering any shape for the triangle, which one has a greater length?), where D, E, and O are the same points as in the above corollary. Before answering this question we need the following lemma which is also a consequence of Bisector Proposition and it is somewhat a complement to it.

Lemma (Bisector Proposition Extended). Suppose that A is a point on the bisector of the angle $\angle xOy$ and B, C are two points on the arms Ox and Oy, respectively with $\angle OBA = \alpha$ and $\angle OCA = \beta$. Then $AB > AC$ if, and only if, either $\alpha > \beta$ with $\alpha + \beta > 180°$ or $\alpha < \beta$ with $\alpha + \beta < 180°$.

Proof. Draw any configuration which satisfies our assumption. Let us assume that $AB > AC$, then by Bisector Proposition $\alpha \neq \beta$ and $\alpha + \beta \neq 180°$. Take the point B' to be the symmetric point of the point B with respect to the bisector of the angle $\angle xOy$. Now either $\alpha > \beta$, in which case, B' manifestly lies on Oy between O and C (note, $\angle OB'A = \angle OBA = \alpha > \beta = \angle OCA$). Consequently in the triangle $\triangle AB'C$, $AB' = AB > AC$ implies that $\beta > 180° - \alpha$, i.e., we have $\alpha > \beta$ with $\alpha + \beta > 180°$. Or we may have $\alpha < \beta$, in which case, B' cannot lie between O and C. Consequently in the triangle $\triangle ACB'$, $AB' = AB > AC$

implies that $180° - \beta > \alpha$ and we are done. The converse is evident.

The following corollary which is an immediate consequence of the previous lemma settles the above question. Corollary 4, which is also clearly a consequence of the next corollary (it's briefly observed), together with this corollary, give a nontrivial universal fact about any triangle. Motivated by this, we cannot help reciting a comment, with emphasis, as in [11, p. 49], that when it comes to deducing results in mathematics just from the definition of an object, nothing can hold a candle to the triangle.

Corollary 5. In a triangle $\triangle ABC$ let the bisectors of angles $\angle B$, $\angle C$, intersect the opposite sides at D and E, respectively, and take O to be the intersection of these bisectors. Then $OE > OD$ if, and only if, either $\angle A < 60°$ with $\angle C < \angle B$ or $\angle A > 60°$ with $\angle C > \angle B$.

Proof. Let us put $\angle AEO = \alpha$ and $\angle ADO = \beta$. Then clearly $\angle A + \angle DOE = 90° + \frac{3\angle A}{2}$. Thus in the quadrilateral $AEOD$ it is manifest that $\angle A < 60°$ (resp., $\angle A > 60°$) if and only if $\alpha + \beta > 180°$ (resp., $\alpha + \beta < 180°$). We also notice that $\alpha > \beta$ (resp., $\alpha < \beta$) if and only if $\angle B > \angle C$ (resp., $\angle B < \angle C$). Finally, it remains to invoke the above lemma to complete the proof.

Remark. It goes without saying that, in the above corollary, $OD > OE$ if and only if $\angle A < 60°$ with $\angle B < \angle C$ or $\angle A > 60°$ with $\angle B > \angle C$. Considering this, we should remind the reader that Corollary 4, is now an immediate consequence of the above corollary, too. We may also recall that in any triangle $\triangle ABC$, $\angle B > \angle C$ if, and only if, the length of the bisector of $\angle B$ is shorter than that of the bisector of $\angle C$. In case $\angle A \leq 60°$ the latter well-known fact is a consequence of our above observations (note, let $\triangle ABC$ be a triangle with $\angle B > \angle C$ and consider any related configuration to Corollaries 4, 5. Then $BD = BO + OD$ and $CE = CO + OE$, where clearly $BO < CO$ and whenever $\angle A \leq 60°$, then in view of Corollaries 4, 5, we have $OD \leq OE$, which implies that $BD < CE$ and we are done). It is worth noting that although the validity of $BD = BO + OD < CO + OE = CE$ does not depend on the measure of $\angle A$. However, one should emphasize that in contrast to the previous case that $BO < CO$, $OD \leq OE$ and consequently $BD = BO + OD < CO + OE = CE$, the case $\angle A > 60°$, reveals a new information in this regard, namely, $BO < CO$, and although $OD > OE$, but still of course $BD = BO + OD < CE = CO + OE = CE$.

After all these discussions and recollection about Bisector Proposition and some of its consequences, everyone is expected to clearly comprehend the proposition, remember it forever; and use it as naturally as the classical fact about Bisectors, like the time we were dealing with geometric facts in our school days. In particular, as a consequence of this proposition, everyone should also be certain visually, that the Morley triangle of a triangle $\triangle ABC$ is equilateral if and only if $\angle 1 = \angle 4$, $\angle 2 =$

$\angle 5$, and $\angle 3 = \angle 6$, see Figure (a) (i.e., anyone with some adequate knowledge of Euclidean geometry including Bisector Proposition could have easily guessed a natural equivalent form of Morley's Theorem). Therefore, one may claim Morley's Theorem is, in essence, a fact related to properties of bisectors. In particular, it is a consequence of a missing property of bisectors, i.e., Bisector Proposition. Let us, in what follows, give some more justifications for the previous claim: Manifestly every trisector of a triangle is, in fact, a bisector of an angle in any configuration related to Morley's Theorem (indeed, of the two trisectors of an angle, $\angle A$ say, in the triangle $\triangle ABC$, clearly the one adjacent to side AB (resp., AC) is the bisector of the angle whose arms are AB (resp., AC) and the other trisector, respectively). Moreover every vertex of the Morley's triangle also lies on the bisector of the angle formed by the intersection of a pair of trisectors, whose intersection is not that vertex. Considering all these evident observations and the natural and simple Bisector Proposition, including its consequences, shouldn't we ask, **where is the mystery, then?**

References

[1] E. D. J. Barbeau, M. S. Klamkin, and W. O. J. Moser, *Five Hundred Mathematical Challenges*, MAA, 1995.

[2] A. Bremner, J. R. Goggins, M. J. T. Guy, and R. K. Guy, On rational Morley triangle, *Acta Arithmetica*, XCIII.2 (2000) 177–187.

[3] A. Cain, Deus ex Machina and the Aesthetics of proofs, *The Mathematical Intelligencer*, 32 (2010), No. 3, 7–11.

[4] J. M. Child, Proof of "Morley's theorem", *Math. Gazzette*, 11 (1923) 171.

[5] A. Connes, A new proof of Morley's theorem, les relations entre les mathematiques et la physique theorique, *Hautes Sci., Bures*, 1998, pp. 43-46, MR 99m:51027.

[6] J. Conway, On Morley's Trisector Theorem, *The Mathematical Intelligencer*, 36(3) (2014) 3.

[7] R. L. Francis, Modern mathematical milestone: Morley's mystery, *Missouri Journal of Mathematical Sciences*, 14 (2002), 1–3.

[8] I. Gorjian, O. A. S. Karamzadeh, and M. Namdari, Morley's theorem is no longer mysterious!, *The Mathematical Intelligencer*, 37 (2015) No.4, 6–7.

[9] R. K. Guy, The lighthouse theorem, Morley & Malfatti-a budget of paradoxes, *Amer. Math. Monthly*, 114 (2007) 97–14.

[10] Y. Hashimoto, A short proof of Morley's theorem, *Elem. Math.*, 62 (2007) 121–121. 0013-6018/07/030121-1

[11] O. A. S. Karamzadeh, The Mathematics of Mathematics Houses (The Snaky Connection), *The Mathematical Intelligencer*, 34 (2012) No. 4, 46–52.

[12] O. A. S. Karamzadeh, O. A. S. Karamzadeh, Is John Conway's proof of Morley's theorem the simplest and free of a deus ex machina? *The Mathematical Intelligencer*, 36 (2014) No.3, 4–7.

[13] Spiridon A. Kuruklis, Trisectors like bisectors with equilaterals instead of points, *CUBO A Mathematical Journal*, 16, No 01, (71110); June 2014.

[14] A. Letac, Solution (Morley's triangle), problem No. 490 [Sphinx:revue mensuelle des questions recreatives, Brussels, 8(1938), 106] Sphinx, 9 (1939), 46.

[15] F. Morley and F. V. Morley, *Inversive Geometry*, Ginn, Boston, 1933. (Reissued by Chelsea, Bronx, New York, 1954 and by Dover Books on Mathematics, 2013).

[16] M. T. Navansiengar, Solution to Morley's problem, *Mathematical Questions and Solutions, from Edu. Times*, New Series 15 (1909) 45.

[17] D. J. Newmann, The Morley's miracle, *The Mathematical Intelligencer*, 18 (1996) No.1, 31–32.

[18] C. O. Oakley and J. C. Baker, The Morley trisector theorem, *Amer. Math. Monthly*, 85 (1978) 737–745.

[19] R. Penrose, Morley's trisector theorem, *Eureka*, 16 (1953) 6–7.
[20] A. H. Schoenfeld, *Mathematical Problem Solving*, Academy Press, 1985.
[21] B. Stonebridge, A simple geometric proof of Morley's trisector theorem, *Applied Probability Trust*, 2009.
[22] T. Tao, *Solving Mathematical Problems, A Personal Perspective*, Oxford University Press, 2006.
[23] G. Wanner, Elementare Beweise des Satzes von Morley, *Elem. Math.*, 59 (2009) 144–150.
[24] R. J. Webster, Morley's triangle theorem, *Math. Mag.*, 43 (1970) 209–210.

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