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A Synthetic Proof of the Equality of Iterated Kiepert Triangles $\mathcal{K}(\phi,\psi)=\mathcal{K}(\psi,\phi)$

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Abstract. We use Miquel's Theorem to prove the equality $\mathcal{K}(\phi,\psi)=\mathcal{K}(\psi,\phi)$ of iterated Kiepert triangles.

1. Introduction

The Kiepert triangle $\mathcal{K}(\phi)=A_\phi B_\phi C_\phi$ is the triangle formed by the apices of isosceles triangles with base angles $-\frac{\pi}{2}<\phi<\frac{\pi}{2}$ erected on sides BC, AC, and BC of triangle ABC respectively. The iterated Kiepert triangle $\mathcal{K}(\phi,\psi)$ is found by subsequently erecting isosceles triangles with base angles ψ to $\mathcal{K}(\phi)$. Paul Yiu and the author showed by calculating homogeneous barycentric coordinates that $\mathcal{K}(\phi,\psi)=\mathcal{K}(\psi,\phi)$ ([2], section 3). In this paper we will show that the result can be acquired by synthetic means, by application of Miquel's Theorem.

2. Application of Miquel's Theorem

We recall Miquel's theorem, in the wording of [1, Theorem 4], for instance, which contains the well known corollaries added to the original theorem [3].

Theorem 1 (Miquel). *Let $A_1B_1C_1$ be a triangle inscribed in triangle ABC. There is a pivot point P such that $A_1B_1C_1$ is the image of the pedal triangle of P after a rotation about P followed by a homothety with center P. All inscribed triangles directly similar to $A_1B_1C_1$ have the same pivot point.*

It is well known that this theorem can be proven syntheticly.

A consequence of this theorem is that if two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are directly similar, then each triangle $A_3B_3C_3$ with $A_3\in A_1A_2$, $B_3\in B_1B_2$, and $C_3\in C_1C_2$ such that the ratios of directed distances fulfill

$$A_1A_3 : A_3A_2 = B_1B_3 : B_3B_2 = C_1C_3 : C_3C_2$$

is directly similar to these as well, as A_1A_2, B_1B_2, and C_1C_2 bound a triangle.

Remark. This result is valid for general similar figures and is often referred to as "Fundamental Theorem of Directly Similar Figures".

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Now we consider triangle ABC, with Kiepert vertices A_ϕ, C_ϕ, A_ψ, C_ψ and iterated Kiepert vertices $B_{\phi,\psi}$ and $B_{\phi,\phi}$. M_a, M_c, and $M_{b,\phi}$ are the midpoints of BC, AB, and $A_\phi C_\phi$ respectively. See Figure 1. Triangles $A_\phi BC$ and $C_\phi AB$ are directly similar, so triangle $M_{b,\phi}M_cM_a$ is directly similar to these and hence an isosceles triangle with base angle ϕ as well. So triangles $M_aM_{b,\phi}M_c$ and $A_\phi B_{\phi,\phi}C_\phi$ are directly similar, and clearly

$$M_aA_\psi : A_\psi A_\phi = M_{b,\phi}B_{\phi,\psi} : B_{\phi,\psi}B_{\psi,\phi} = M_cC_\psi : C_\psi C_\phi.$$

This shows that triangle $A_\psi B_{\phi,\psi}C_\psi$ is also isosceles with base angle ϕ. We have derived that $B_{\phi,\psi} = B_{\psi,\phi}$. With similar reasoning for the A- and C-vertices we conclude the proof that $\mathcal{K}(\phi,\psi) = \mathcal{K}(\psi,\phi)$.

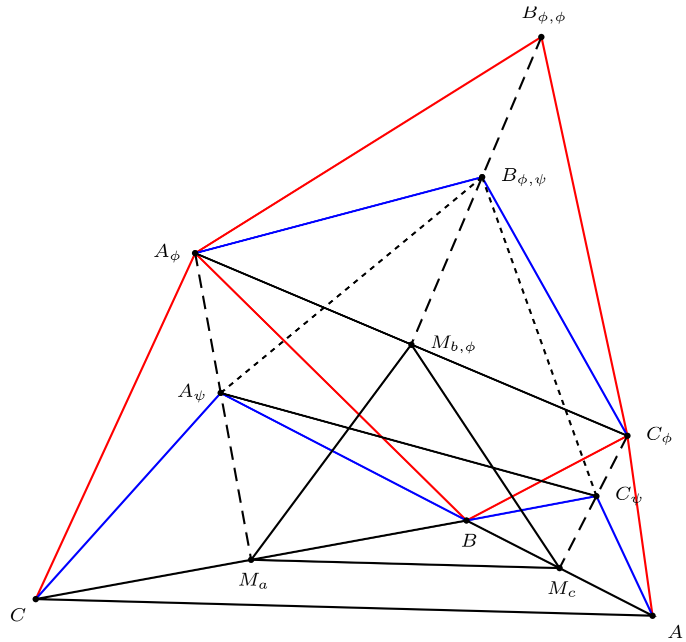

Figure 1

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