Forum Geometricorum
Volume 18 (2018) 349–351.

FORUM GEOM
ISSN 1534-1178

Orthopoles and Variable Flanks

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Abstract. We extend Zaharinov's result on the perspector of the triangle of flank line orthopoles to variable flanks. The locus of perspectors is the Kiepert hyperbola.

1. Introduction

Zaharinov [3] has studied the orthopoles of the a-, b-, c-sides of the of the A-, B-, and C-flanks respectively. If we attach squares to the sides of triangle ABC we find at each vertex of ABC a flank triangle [2]. He found that these orthopoles form a triangle perspective with ABC, with the Vecten point as perspector. In this paper we extend the results to variable flanks, replacing the attached squares by similar rectangles, which Čerin [1] used to find various loci. We show that the orthopoles of variable flanks form triangles perspective with ABC, the locus of perspectors being the Kiepert hyperbola.

2. Orthopoles of flank lines

Consider rectangles ABC_bC_a, BCA_cA_b, and CAB_aB_c attachted to the sides of triangle ABC and satisfying $\angle BAC_b = \angle CBA_c = \angle ACB_a = \phi$. We will calculate barycentrics for the orthopole of line A_bC_b, the B-flank line (see Figure 1).

We have that

$$A_b = (-a^2 : S_C + S_\phi : S_B),$$
$$C_b = (S_B : S_A + S_\phi : -c^2).$$

Lines perpendicular to A_bC_b are parallel to the B-median of ABC as the orthocenter and centroid are friends (see [2]). So these lines meet the line at infinitiy $\mathcal{L}^\infty$ in the point $(1 : -2 : 1)$.

The perpendicular ℓ_A through A to A_bC_b hence has equation $\ell_A : \quad y + 2z = 0$, while the equation for A_bC_b is given by

$$(S^2 + (c^2 + S_B)S_\phi)x + S^2 y + (S^2 + (a^2 + S_B)S_\phi)z = 0.$$

If we denote by $\overline{\phi}$ the complement of ϕ, the latter equation can be rewritten as

$$(S_B + c^2 + S_{\overline{\phi}})x + S_{\overline{\phi}}y + (S_B + a^2 + S_{\overline{\phi}})z = 0,$$

Publication Date: November 19, 2018. Communicating Editor: Paul Yiu.

noting that $S^2 = S_{\phi\overline{\phi}}$.
So the point A' where ℓ_A and A_bC_b intersect has coordinates

$$A' = (-S_B - a^2 + S_{\overline{\phi}} : \ -2(S_B + c^2 + S_{\overline{\phi}}) : \ S_B + c^2 + S_{\overline{\phi}}).$$

Similarly, if ℓ_C is the perpendicular through C to A_bC_b, then the point C' where ℓ_C and A_bC_b meet has coordinates

$$C' = (S_B + a^2 + S_{\overline{\phi}} : \ -2(S_B + a^2 + S_{\overline{\phi}}) : \ -S_B - c^2 + S_{\overline{\phi}}).$$

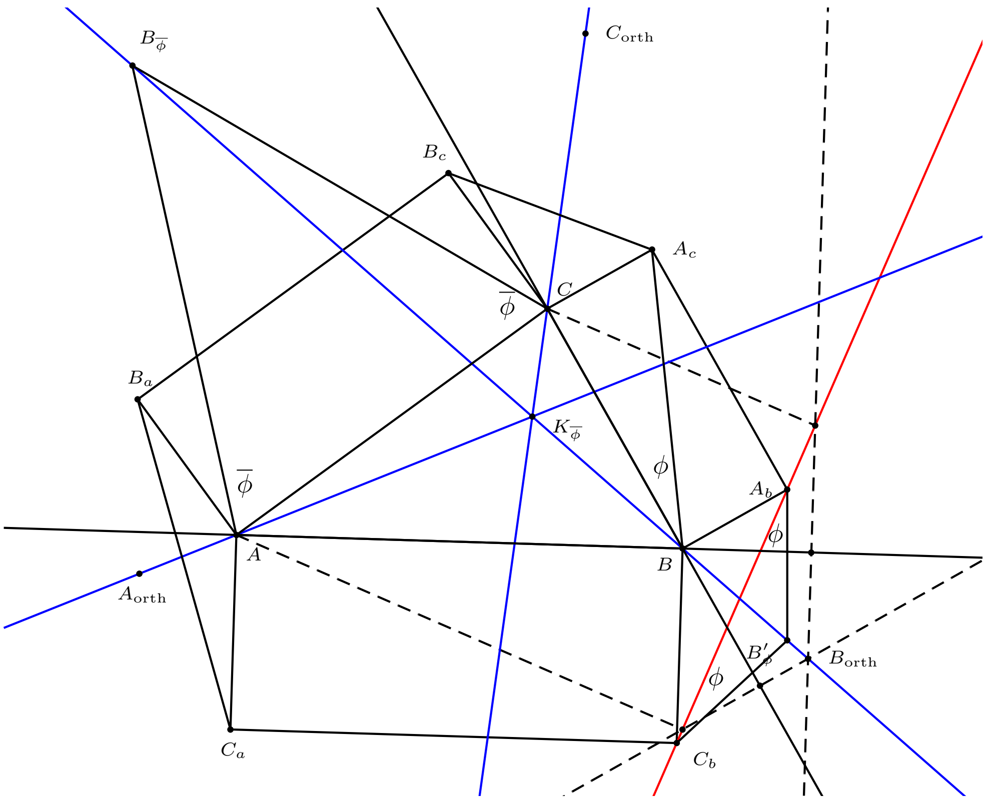

Figure 1.

Now the line through A' perpendicular to BC is given by

$$\begin{aligned}&(S_B + a^2)(S_B + c^2 + S_{\overline{\phi}})x + ((S_B + a^2)S_{\overline{\phi}} + S^2)y \\ &\quad + ((S_B + a^2)S_{\overline{\phi}} + S_B(S_C + 3a^2) + a^2(b^2 + c^2))z = 0.\end{aligned}$$

With a similar result for the line through C' perpendicular to AB we find as point of intersection for these two lines the orthopole of the B-flank line

$$\begin{aligned} B_{\text{orth}} = &\Big(S^2(2a^2 - b^2 + 2c^2)(S_C + S_{\overline{\phi}}) \\ &: \ -4S^2(2a^2 - b^2 + 2c^2)(a^2 + c^2 + S_{\overline{\phi}}) \\ &: \ S^2(2a^2 - b^2 + 2c^2)(S_A + S_{\overline{\phi}})\Big) \\ = &\ (S_C + S_{\overline{\phi}} : \ -4(a^2 + c^2 + S_{\overline{\phi}}) : \ S_A + S_{\overline{\phi}}). \end{aligned}$$

By symmetry this shows that the triangle of the three orthopoles of the flank lines is perspective to ABC, the Kiepert perspector $K_{\overline{\phi}}$ being the perspector. For variable flanks the line will hence run through the Kiepert hyperbola. As $K_{\overline{\phi}}$ and K_ϕ are friends, we know that the line connecting B, $K_{\overline{\phi}}$, and B_{orth} will also pass through the apices of isosceles triangles erected on AC and A_bC_b with base angles $\overline{\phi}$ and ϕ respectively. Naturally, the orthopole of BC and the Kiepert ϕ-perspector, both with respect to the B-flank, join this line, to complete the friendly symmetry.

References

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