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# Simple Proofs of Feuerbach's Theorem and Emelyanov's Theorem

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**Abstract**. We give simple proofs of Feuerbach's Theorem and Emelyanov's Theorem with the same idea of using the anticomplement of a point and homogeneous barycentric coordinates.

## 1. Proof of Feuerbach's Theorem

Feuerbach's Theorem is known as one of the most important theorems with many applications in elementary geometry; see [1, 2, 3, 5, 6, 8, 9, 10, 12].

**Theorem 1** (Feuerbach, 1822)**.** *In a nonequilateral triangle, the nine-point circle is internally tangent to the incircle and is externally tangent to the excircles.*

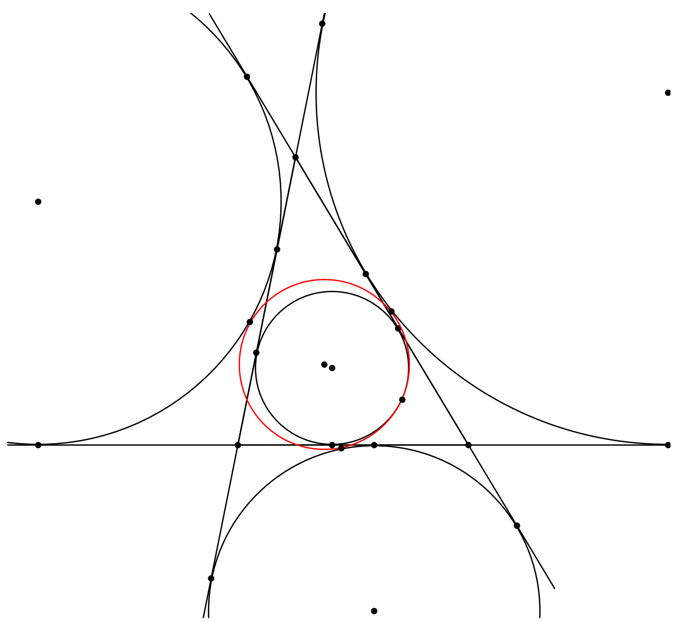

Figure 1

We establish a lemma to prove Feuerbach's Theorem.

**Lemma 2.** *The circumconic* $p^2yz + q^2zx + r^2xy = 0$ *is tangent to line at infinity* $x + y + z = 0$ *if and only if*

$$(p + q + r)(-p + q + r)(p - q + r)(p + q - r) = 0$$

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*and the point of tangency is the point* $Q = (p : q : r)$ *or* $Q_a = (-p : q : r)$ *or* $Q_b = (p : -q : r)$ *or* $Q_c = (p : q : -r)$ *according to which factor of the above product is zero.*

*Proof.* We shall prove that the system

$$\begin{cases} p^2yx + q^2zx + r^2xy = 0 \\ x + y + z = 0 \end{cases}$$

has only one root under the above condition.

Substituting $x = -y - z$, we get

$$p^2yz - (q^2z + r^2y)(y + z) = 0.$$

The resulting quadratic equation for $y/z$ is

$$r^2y^2 + (q^2 + r^2 - p^2)yz + q^2z^2 = 0.$$

This quadratic equation has discriminant

$$\begin{aligned} D = &\ (q^2 + r^2 - p^2)^2 - 4(qr)^2 \\ = &\ -(p + q + r)(-p + q + r)(p - q + r)(p + q - r). \end{aligned}$$

Thus (1) has only one root if and only if the discriminant $D = 0$.

If $\varepsilon_1 p + \varepsilon_2 q + \varepsilon_3 r = 0$ for $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3 = \pm 1$, then $D = 0$, which means that the circumconic $p^2yz + q^2zx + r^2xy = 0$ or $\frac{p^2}{x} + \frac{q^2}{y} + \frac{r^2}{z} = 0$ has a double point on the line at infinity and that the point $Q = (\varepsilon_1 p,\ \varepsilon_2 q,\ \varepsilon_3 r)$ lies on the line at infinity and also on the circumconic because $\dfrac{p^2}{\varepsilon_1 p} + \dfrac{q^2}{\varepsilon_2 q} + \dfrac{r^2}{\varepsilon_3 r} = \varepsilon_1 p + \varepsilon_2 q + \varepsilon_3 r = 0.$ Hence this point $Q$ is the double point on the line at infinity of the circumconic.

That leads to a proof of the lemma. □

*Remark.* The conic which is tangent to the line at infinity must be a parabola. Thus our lemma is exactly a characterization of the circumparabola of a triangle.

*Proof of Feuerbach's Theorem.* Let $G$ be the centroid of triangle $ABC$, and the incircle of $ABC$ touch $BC$, $CA$, and $AB$ at $D$, $E$, and $F$ respectively. The homothety $\mathcal{H}(G, -2)$ transforms a point $P$ to its anticomplement $P'$, which divides $PG$ in the ratio $PG : GP' = 1 : 2$ (see [11]). We consider the anticomplement of the points $D$, $E$, and $F$, which are $X$, $Y$, and $Z$ respectively (see Figure 2). Under the homothety $\mathcal{H}(G, -2)$, the nine-point circle of $ABC$ transforms to circumcircle $(\Gamma)$ of $ABC$. Thus, it is sufficient to show that the circumcircle $(\omega)$ of triangle $XYZ$ is tangent to $(\Gamma)$.

Let $M$ be the midpoint of $BC$. Then $\mathcal{H}(G, -2)$ maps $M$ to $A$. Thus, $AX = 2DM = |b - c|$, and $AX$ is also tangent to $(\omega)$. We deduce that the power of $A$ with respect to $(\omega)$ is $AX^2 = (b - c)^2$. Similarly, the power of $B$ with respect to $(\omega)$ is $(c - a)^2$, and the power of $C$ with respect to $(\omega)$ is $(a - b)^2$.

Using the equation of a general circle in [12], we have the equation of $(\omega)$ as follows:

$$a^2yz + b^2zx + c^2xy - (x + y + z)((b - c)^2x + (c - a)^2y + (a - b)^2z) = 0.$$

Hence, the line $L:\ (b-c)^2x+(c-a)^2y+(a-b)^2z=0$ is the radical axis of $(\Gamma)$ and $(\omega)$.

Analogously with the excircles, we can find that the radical axes of the circumcircle with the homothetic of the excircles under $\mathcal{H}(G,-2)$ are the lines

$$\begin{aligned}
L_a:&\quad (b-c)^2x+(c+a)^2y+(a+b)^2z=0,\\
L_b:&\quad (b+c)^2x+(c-a)^2y+(a+b)^2z=0,\\
L_c:&\quad (b+c)^2x+(c+a)^2y+(a-b)^2z=0.
\end{aligned}$$

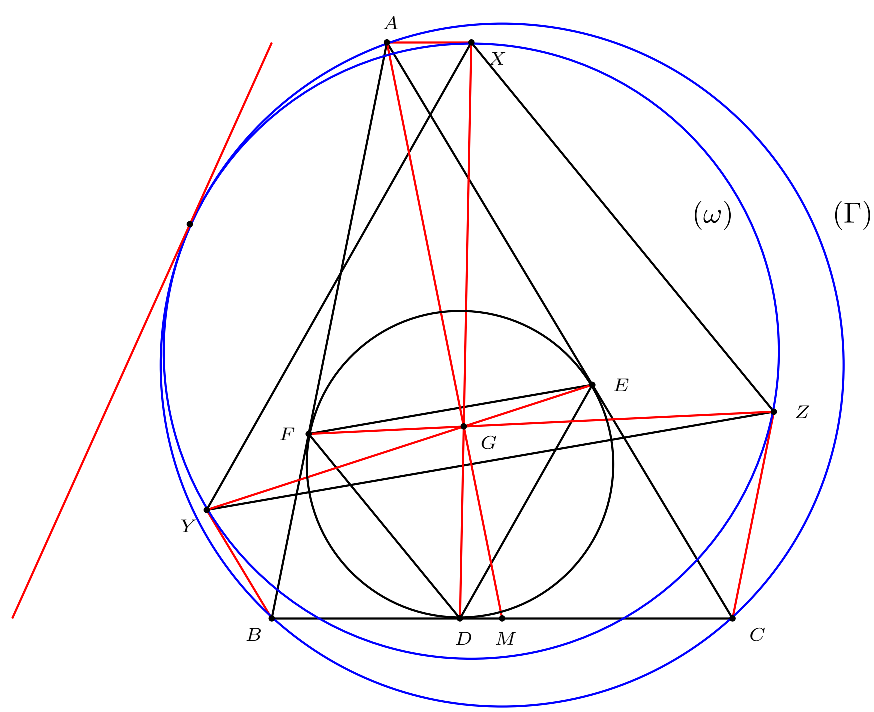


Figure 2

In order to prove Feuerbach's theorem, we shall prove that the lines $L$, $L_a$, $L_b$, and $L_c$ are tangent to the circumcircle $(\Gamma)$ of $ABC$: The isogonal conjugate of $L$, $L_a$, $L_b$, and $L_c$ are respectively the conics

$$\begin{aligned}
LL:&\quad (a(b-c))^2yz+(b(c-a))^2zx+(c(a-b))^2xy=0,\\
LL_a:&\quad (a(b-c))^2yz+(b(c+a))^2zx+(c(a+b))^2xy=0,\\
LL_b:&\quad (a(b+c))^2yz+(b(c-a))^2zx+(c(a+b))^2xy=0,\\
LL_c:&\quad (a(b+c))^2yz+(b(c+a))^2zx+(c(a-b))^2xy=0.
\end{aligned}$$

Using Lemma 2, we easily check that the conics $LL$, $LL_a$, $LL_b$, and $LL_c$ are tangent to the line at infinity $x+y+z=0$, which is the isogonal conjugate of circumcircle $(\Gamma)$.

In particular, for $LL$, the point of tangency is the point $Q=(a(b-c):b(c-a):c(a-b))$, and in order to find the Feuerbach point (which is the contact point of nine-point circle with the incircle), we find the isogonal conjugate of $Q$. This is $\left(\frac{a}{b-c}:\ \frac{b}{c-a}:\ \frac{c}{a-b}\right)$, the anticomplement of $X_{11}$.

This completes the proof.

## 2. Proof of Emelyanov's Theorem

Continuing with the idea of using the anticomplement of a point [11] and barycentric coordinates [12], we give a simple proof for Lev Emelyanov's theorem [4, 7].

**Theorem 3** (Emelyanov, 2001)**.** *The circle passing through the feet of the internal bisectors of a triangle contains the Feuerbach point of the triangle.*

**Lemma 4.** *Let $ABC$ be a triangle and the point $P$ $(x:y:z)$ in homogeneous barycentric coordinates, with cevian triangle $A'B'C'$. If $M$ is the midpoint of $BC$, then signed length of $MA'$ is*

$$MA'=\frac{z-y}{2(y+z)}BC.$$

*Proof.* Because $A'B'C'$ is the cevian triangle of $P$, $A'=(0:y:z)$. Using signed lengths of segments, we have $BA'=\frac{z}{y+z}BC$ and $CA'=\frac{y}{y+z}CB$. Therefore, we get the signed length of $MA'$:

$$MA'=\frac{BA'+CA'}{2}=\frac{1}{2}\left(\frac{z}{y+z}BC+\frac{y}{y+z}CB\right)=\frac{z-y}{2(y+z)}BC.$$

We are done. □

*Proof of Emelyanov's Theorem.* In the triangle ABC, let $A_1$, $B_1$, and $C_1$ be the feet of the internal bisectors of the angles $A$, $B$, and $C$ respectively. Let $(\gamma)$ be the circumcircle of the triangle $A_1B_1C_1$, we must prove that $(\gamma)$ contains the Feuerbach point $F_{\mathrm{e}}$.

Let the circle $(\gamma)$ meet the sides $BC$, $CA$ and $AB$ again at $A_2$, $B_2$ and $C_2$ respectively. From Carnot's theorem [12], we have that the triangle $A_2B_2C_2$ is the cevian triangle of a point $I^*=(x:y:z)$ such that $BC_1\cdot BC_2=BA_1\cdot BA_2$ and $CA_1\cdot CA_2=CB_1\cdot CB_2$. From these, we get

$$\frac{ac}{a+b}\cdot\frac{xc}{x+y}=\frac{za}{y+z}\cdot\frac{ca}{b+c}\quad\text{and}\quad\frac{ba}{b+c}\cdot\frac{ya}{y+z}=\frac{xb}{z+x}\cdot\frac{ab}{c+a}$$

or

$$\begin{array}{lll} a_1=a(a+b), & b_1=(a-c)(a+b+c), & c_1=-c(b+c),\\ a_2=-a(c+a), & b_2=b(b+c), & c_2=(b-a)(a+b+c). \end{array}$$

We have the system

$$\begin{cases} a_1yz + b_1zx + c_1xy = 0 \\ a_2yz + b_2zx + c_2xy = 0 \end{cases}$$

$$\Longleftrightarrow \frac{yz}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{zx}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{xy}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

or

$$(x : y : z) = \left( \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right)^{-1}$$

This gives

$$\begin{aligned} I^* = \ & \big(abc + (a+b+c)(-a^2+b^2+c^2) \\ & : \ abc + (a+b+c)(a^2-b^2+c^2) \\ & : \ abc + (a+b+c)(a^2+b^2-c^2)\big)^{-1}. \end{aligned}$$

$I^*$ is also the cyclocevian of I which is the point $X(1029)$ [5].

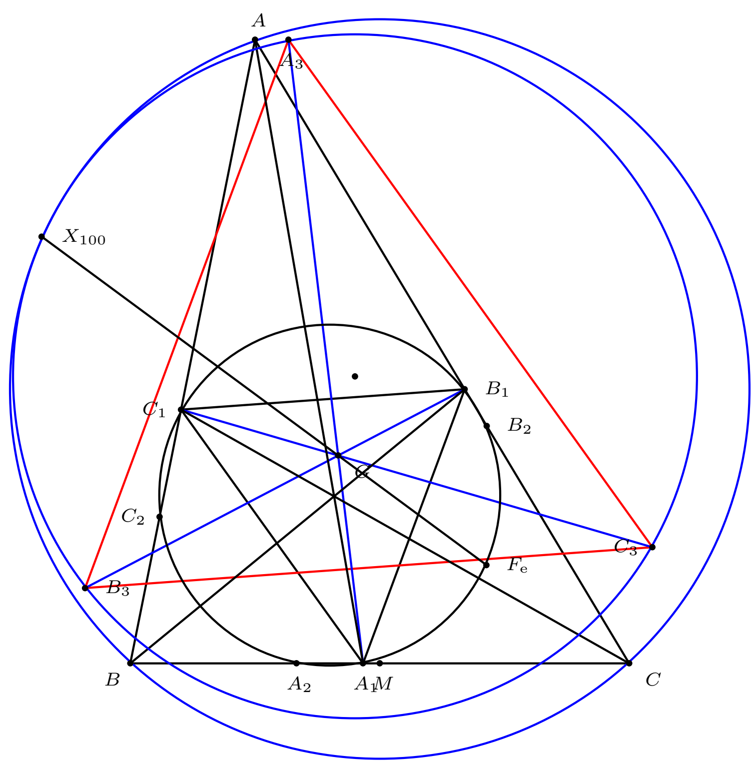


Figure 3

Let $M$ be the midpoint of $BC$. Using Lemma 4, the signed length of $MA_1$ is

$$MA_1 = \frac{c-b}{2(b+c)}BC.$$

and the signed length of $MA_2$ is

$$MA_2 = \frac{\left(\frac{1}{abc+(a+b+c)(a^2+b^2-c^2)} - \frac{1}{abc+(a+b+c)(a^2-b^2+c^2)}\right)}{2\left(\frac{1}{abc+(a+b+c)(a^2-b^2+c^2)} + \frac{1}{abc+(a+b+c)(a^2+b^2-c^2)}\right)}BC.$$

An easy simplification leads to

$$MA_2 = \frac{(c^2-b^2)(a+b+c)}{2a(c+a)(a+b)}BC.$$

Let $G$ be the centroid of $ABC$: We consider the anticomplement of the points $A_1$, $B_1$, and $C_1$, which are the points $A_3$, $B_3$, and $C_3$. Under the homothety $\mathcal{H}(G,-2)$, in order to prove that $(\gamma)$ contains the Feuerbach point of $ABC$, we shall show that the circumcircle $(\Omega)$ of the triangle $A_3B_3C_3$ contains the anticomplement of the Feuerbach point. Indeed, as in our above proof of Feuerbach's theorem, we showed that anticomplement of Feuerbach point is

$$X(100) = \left(\frac{a}{b-c} \,:\, \frac{b}{c-a} \,:\, \frac{c}{a-b}\right).$$

Because the anticomplement of $M$ is $A$, the power of $A$ with respect to circumcircle of triangle $A_3B_3C_3$ is

$$p = 2MA_1{\cdot}2MA_2 = \frac{(c-b)}{(b+c)}BC{\cdot}\frac{(c^2-b^2)(a+b+c)}{a(c+a)(a+b)}BC = \frac{a(a+b+c)(b-c)^2}{(c+a)(a+b)}.$$

Similarly, the powers of $B$ and $C$ with respect to circumcircle of triangle $A_3B_3C_3$ are

$$q = \frac{b(a+b+c)(c-a)^2}{(b+c)(b+a)} \quad \text{and} \quad r = \frac{c(a+b+c)(a-b)^2}{(c+a)(b+c)}$$

respectively.

Using the equation of a general circle in [12] again, we have the equation of $(\Omega)$:

$$a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0.$$

Using the coordinates of $X(100)$, we easily check the expression

$$a^2yz+b^2zx+c^2xy = a^2\cdot\frac{b}{c-a}\cdot\frac{c}{a-b}+b^2\cdot\frac{c}{a-b}\cdot\frac{a}{b-c}+c^2\cdot\frac{a}{b-c}\cdot\frac{b}{c-a} = 0.$$

and

$$\begin{aligned}
px+qy+rz &= p\cdot\frac{a}{b-c}+q\cdot\frac{b}{c-a}+r\cdot\frac{c}{a-b}\\
&= \frac{a(a+b+c)(b-c)^2}{(c+a)(a+b)}\cdot\frac{a}{b-c}+\frac{b(a+b+c)(c-a)^2}{(b+c)(b+a)}\cdot\frac{b}{c-a}\\
&\quad+\frac{c(a+b+c)(a-b)^2}{(c+a)(b+c)}\cdot\frac{c}{a-b}\\
&= 0.
\end{aligned}$$

Hence, the coordinates of $X(100)$ satisfy the equation of $(\Omega)$. This means that $(\Omega)$ passes through the anticomplement of the Feuerbach point. In other words $(\gamma)$ passes through Feuerbach point. This completes our proof of Emelyanov's theorem.

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