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Circumconics with Asymptotes Making a Given Angle

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Abstract. Given a triangle ABC, we present a construction of the circumhyperbola whose two asymptotes are parallel to the sides of an isosceles triangle on a side of ABC.

Let ABC be a triangle and $0 < \theta < \frac{\pi}{2}$. Call U the point on the perpendicular bisector of BC such that $\angle BUC = 2\theta$ and $\mathcal{H}_{ab}$ the circumhyperbola of ABC with asymptotes parallel to UB and UM.

To construct this hyperbola by five ordinary points, we consider the points D and D' defined as follows:

- Let M be the midpoint BC and S the intersection point of the parallel to BU through C and the parallel to AB through M. Then the line AS intersects the perpendicular bisector of BC at D.
- Let S' be the intersections of the parallel through A to BC and the parallel through C to BU. If the parallel through S' to AB intersects BC at M', then the perpendicular through M' to BC intersects the parallel through A to BC at D'.

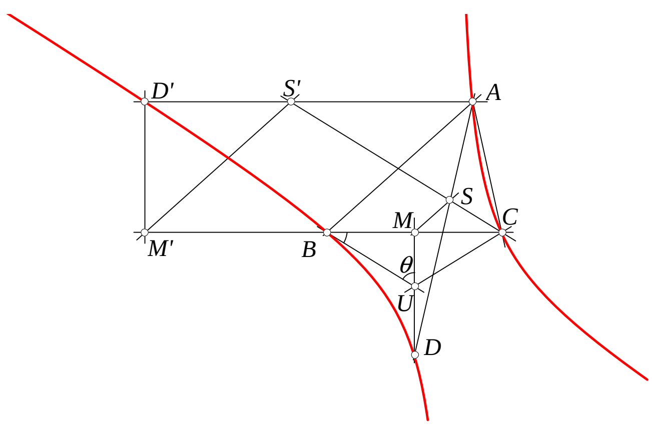

Figure 1

Proposition 1. *The points D, D' lie on $\mathcal{H}_{ab}$.*

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Proof. We have the coordinates

$$D = \left(a^4 : (S_B - S_C)(S_C - S\tan\theta) : -(S_B - S_C)(S_B + S\tan\theta)\right),$$
$$D' = \left(a^2 : S_B - S_C + S\tan\theta : -S_B + S_C - S\tan\theta\right).$$

On the other hand, the hyperbola $\mathcal{H}_{ab}$ has equation

$$a^4yz + S_C\left(S_C - S\tan\theta\right)zx + S_B\left(S_B + S\tan\theta\right)xy = 0.$$

□

Remark. Proposition 1 can also be proved by using Pascal theorem.

Let A_b the perspector of the hyperbola $\mathcal{H}_{ab}$, and A_c the perspector of the hyperbola $\mathcal{H}_{ac}$, defined in a similar way, that is, as the circumhyperbola with asymptotes parallel to UM and UC. Therefore we have the points

$$A_b = \left(a^4 : S_C\left(S_C - S\tan\theta\right) : S_B\left(S_B + S\tan\theta\right)\right),$$
$$A_c = \left(a^4 : S_C\left(S_C + S\tan\theta\right) : S_B\left(S_B - S\tan\theta\right)\right).$$

We also define B_c, B_a and C_a, C_b cyclically.

Proposition 2. *The six points A_b, A_c, B_c, B_a, C_a, C_b, lie on a same conic Γ_θ, whose equation is*

$$\tan^2\theta \cdot (S_Ax + S_By + S_Cz)^2 - S^2 \cdot \left(x^2 + y^2 + z^2 - 2xy - 2xz - 2yz\right) = 0.$$

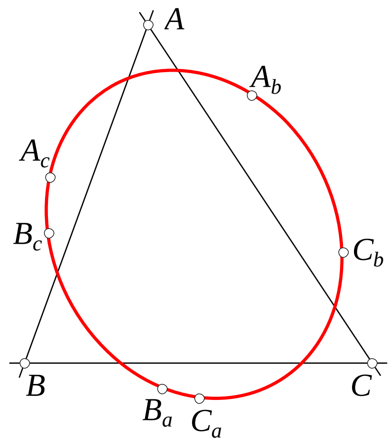

Figure 2

Proposition 3. *The center O_θ of Γ_θ lies on the line GK and the following ratio holds:*

$$\frac{GO_\theta}{O_\theta K} = -\frac{S_\omega^2 \cdot \sin^2\theta}{3S^2},$$

where G and K are the centroid and the symmedian point of ABC.

Proposition 4. *The conic Γ_θ is the locus of perspectors of all circumconics of ABC whose asymptotes make an angle θ.*

Now call XYZ the triangle bounded by lines A_bA_c, B_cB_a, C_aA_b. Surprisingly enough, XYZ does not depend on θ. More specifically, we have the following result.

Proposition 5. *The line A_bA_c trisects the $A-$altitude from the vertex and intersects BC at D', the harmonic conjugate of the feet D of the $A-$altitude with respect to BC. In other words, lines A_bA_c, BC and the sideline of the orthic triangle corresponding to vertex A are concurrent.*

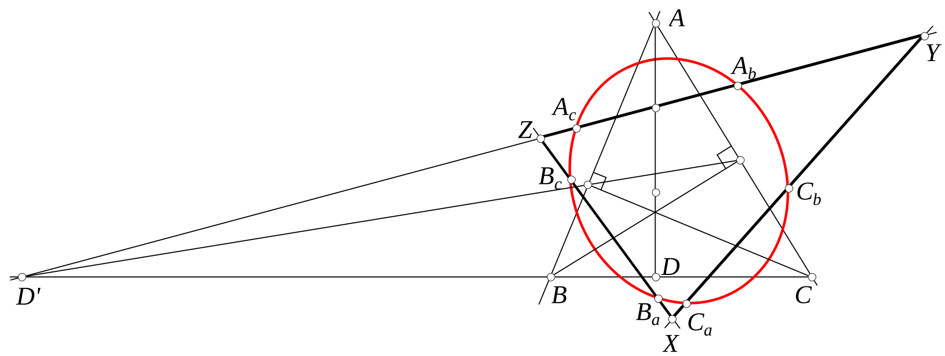

Figure 3

Proposition 6. *The triangle XYZ is perspective at K.*

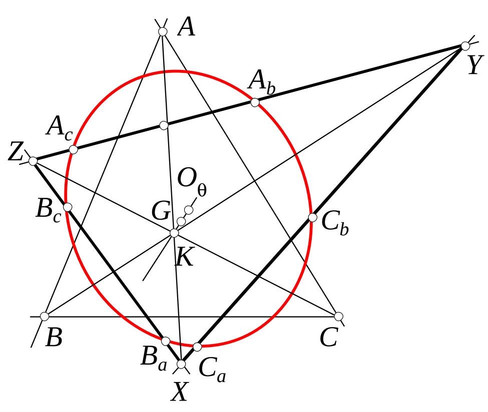

Figure 4

Proposition 7. *The discriminant of Γ_θ is given by the formula*

$$\Delta(\theta) = 2\left(\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2\right)\tan^2\theta - 12S^2.$$

The following figure shows a general view: The point U on the perpendicular bisector of BC has been used to construct the circumhyperbola with perspector A_b having asymptotes parallel to UB and perpendicular to BC. Similarly we construct A_c, B_c, B_a, C_a, C_b lying on the conic Γ_θ. The conic center O_θ of Γ_θ lies on line GK.

Now we take a point P on Γ_θ and construct the conic (hyperbola) with perspector P. The asymptotes of this hyperbola make an angle θ formed by UB and the perpendicular bisector of BC.

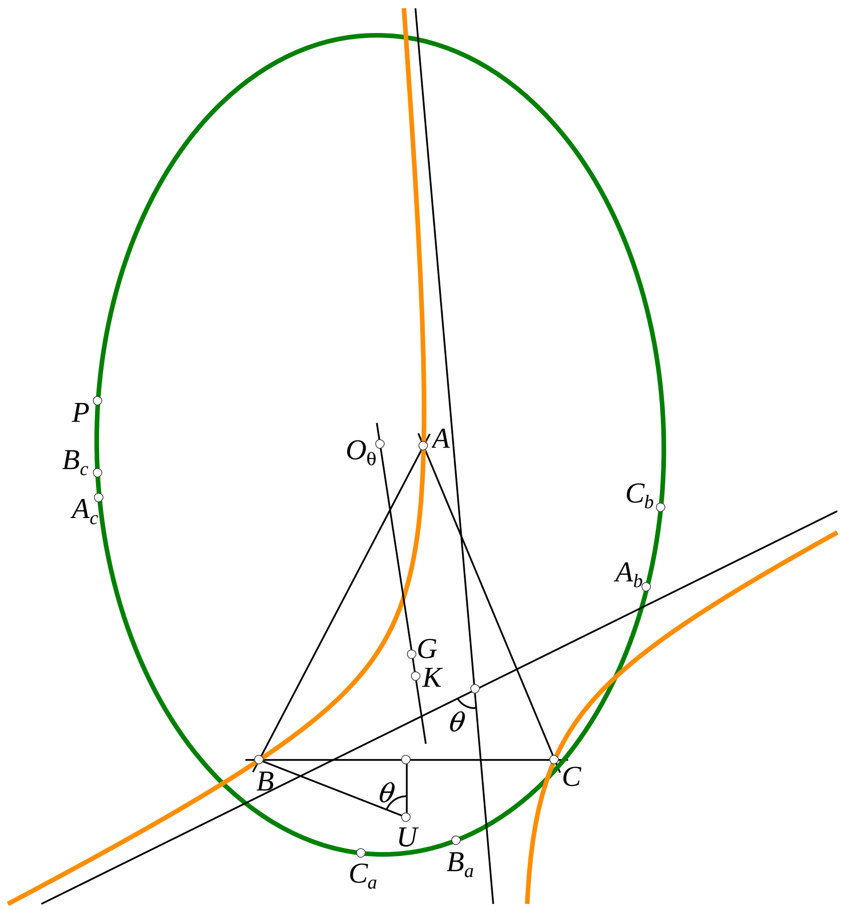

Figure 5

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