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Integral Triangles and Trapezoids Pairs with a Common Area and a Common Perimeter

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Abstract. Using the theory of elliptic curves, we show that there are infinitely many integral right triangle and θ-integral right (resp. isosceles) trapezoid, integral isosceles triangle and θ-integral right (resp. isosceles) trapezoid, Heron triangle and θ-integral right (resp. isosceles) trapezoid pairs with a common area and a common perimeter.

1. Introduction

We say that a Heron (resp. rational) triangle is a triangle with integral (resp. rational) sides and integral (resp. rational) area, and a polygon (sides greater than 3) is integral (resp. rational) if the lengths of its sides are all integers (resp. rational numbers). We call a polygon θ-integral (resp. θ-rational) if the polygon has integral (resp. rational) side lengths, integral (resp. rational) area, and both $\sin\theta$ and $\cos\theta$ are rational numbers.

In 1995, R. K. Guy [6] introduced a problem of Bill Sands, that asked for examples of an integral right triangle and an integral rectangle with a common area and a common perimeter, but there are no non-degenerate such. In the same paper, R. K. Guy showed that there are infinitely many such integral isosceles triangle and integral rectangle pairs. Several authors studied other cases, such as two distinct Heron triangles by A. Bremner [1], Heron triangle and rectangle pairs by R. K. Guy and A. Bremner [2], integral right triangle and parallelogram pairs by Y. Zhang [8], integral right triangle and rhombus pairs by S. Chern [3], integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs by P. Das, A. Juyal and D. Moody [4], Heron triangle-rhombus and isosceles triangle-rhombus pairs by Y. Zhang and J. Peng [9], and rational (primitive) right triangle-isosceles triangle pairs by Y. Hirakawa and H. Matsumura [7].

Now we consider other pairs of geometric shapes having a common area and a common perimeter, such as triangles and trapezoids pairs (see figures 1 and 2). The sides of triangles and trapezoids are given in Table 1, and the area and perimeter of

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triangles and trapezoids are given in Table 2. For θ-integral trapezoid, we may set

$$\sin\theta = \frac{2t}{1+t^2},\ \cos\theta = \frac{1-t^2}{1+t^2},$$

where $0 < t \leq 1$ is a rational number. For $t = 1, \theta = \pi/2$, trapezoid reduces to rectangle, this is the case studied by R. K. Guy [6], thus we only need to consider the case $0 < t < 1$. Using the theory of elliptic curves, we prove

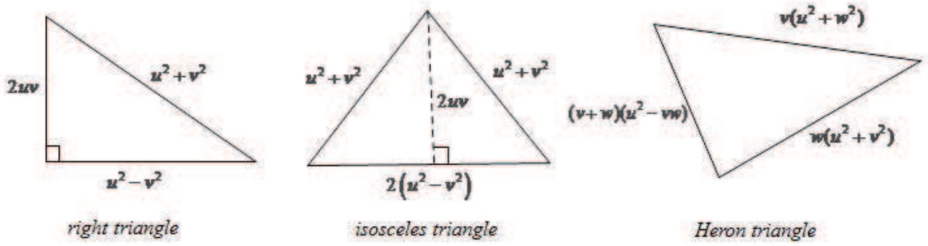

Figure 1. Right triangle, isosceles triangle and Heron triangle

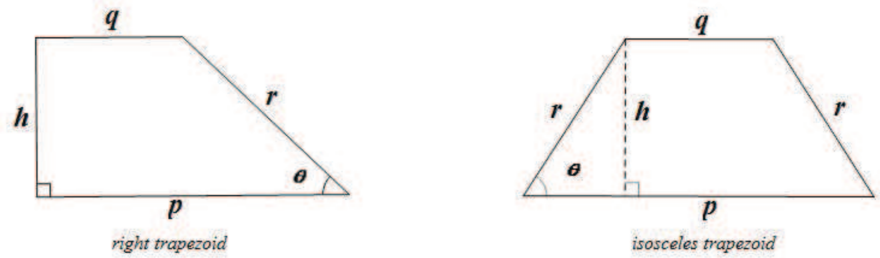

Figure 2. Right trapezoid and isosceles trapezoid

name	sides
right triangle	$(u^2-v^2, 2uv, u^2+v^2)$ $(u > v)$
isosceles triangle	$(u^2+v^2, u^2+v^2, 2(u^2-v^2))$ $(u > v)$
Heron triangle	$((v+w)(u^2-vw), v(u^2+w^2), w(u^2+v^2))$ $(u^2 > vw)$
right trapezoid	(p, q, h, r)
isosceles trapezoid	(p, q, r, r)

Table 1. The sides of triangles and trapezoids

Theorem 1. *There are infinitely many integral right triangle and θ-integral right (resp. isosceles) trapezoid pairs with a common area and a common perimeter.*

name	area	perimeter
right triangle	$uv(u^2-v^2)$	$2u(u+v)$
isosceles triangle	$2uv(u^2-v^2)$	$4u^2$
Heron triangle	$uvw(v+w)(u^2-vw)$	$2u^2(v+w)$
right trapezoid	$\frac{(p+q)h}{2}$	$p+q+h+r$
isosceles trapezoid	$\frac{(p+q)r}{2}\sin\theta$	$p+q+2r$

Table 2. The area and perimeter of triangles and trapezoids

Theorem 2. *There are infinitely many integral isosceles triangle and θ-integral right (resp. isosceles) trapezoid pairs with a common area and a common perimeter.*

Theorem 3. *There are infinitely many Heron triangle and θ-integral right (resp. isosceles) trapezoid pairs with a common area and a common perimeter.*

2. Proofs of the theorems

Proof of Theorem 1. By the homogeneity of the sides of integral right triangle and θ-integral right trapezoid, we can set $u=1$, and v,p,q,h,r to be positive rational numbers. Now we only need to study the rational right triangle and θ-rational right trapezoid pairs with a common area and a common perimeter, then we have

$$\begin{cases} v(1-v^2) = \dfrac{1}{2}(p+q)h, \\ 2(1+v) = p+q+r+h, \\ \dfrac{p-q}{r} = \cos\theta, \\ h = r\sin\theta, \end{cases} \tag{1}$$

where $\sin\theta = 2t/(1+t^2)$ and $\cos\theta = (1-t^2)/(1+t^2)$, with $0<t<1$ a rational number.

Eliminating h,p,q in equation (1), we get

$$\frac{t(t+1)^2r^2 - 2t(t^2+1)(v+1)r - v(v-1)(v+1)(t^2+1)^2}{(t^2+1)^2} = 0.$$

It only needs to consider

$$t(t+1)^2r^2 - 2t(t^2+1)(v+1)r - v(v-1)(v+1)(t^2+1)^2 = 0.$$

If this quadratic equation has rational solutions r, then its discriminant $\Delta_1(r)$ should be a rational perfect square. Let us consider the curve

$$\mathcal{C}_1:\ s^2 = \Delta_1(r) = t(t+1)^2v^3 + t^2v^2 - t(t^2+1)v + t^2,$$

which is a cubic curve with a rational point $P_0 = (v, s) = (1, 2t)$. And the curve $\mathcal{C}_1$ is birationally equivalent to the elliptic curve $\mathcal{E}_1$ given by the equation in Weierstrass form

$$\mathcal{E}_1 :\ Y^2 = X^3 - 27t^2(3t^4+6t^3+7t^2+6t+3)X + 54t^4(3t^2+5t+3)(6t^2+11t+6).$$

The map $\varphi_1 : \mathcal{C}_1 \ni (v, s) \mapsto (X, Y) \in \mathcal{E}_1$ is given by

$$X = 3t(3t^2v + 6tv + t + 3v),\ Y = 27st(t + 1)^2,$$

and the inverse map

$$\varphi_1^{-1} :\ v = \frac{-3t^2X}{9t(t + 1)^2},\ s = \frac{Y}{27t(t + 1)^2}.$$

By the map φ_1, we get the point

$$P = \varphi_1(P_0) = (3t(3t^2 + 7t + 3),\ 54t^2(t + 1)^2),$$

which lies on the elliptic curve $\mathcal{E}_1$. By the group law, we have

$$[2]P = \left(\frac{9t^4 - 6t^2 + 9}{4},\ -\frac{27(t^2 - 4t + 1)(t + 1)^4}{8}\right).$$

Through the inverse map φ_1^{-1}, we get

$$v = \varphi_1^{-1}(x([2]P)) = \frac{(t - 1)^2}{4t},$$

then

$$\begin{aligned} r =& \frac{(t^2 + 1)(t - 1)^2}{8t^2}, \\ p =& -\frac{(t + 1)(t^2 - 4t - 1)}{8t}, \\ q =& \frac{(t + 1)(t^2 + 4t - 1)}{8t^2}, \\ h =& \frac{(t - 1)^2}{4t}. \end{aligned}$$

Hence, the rational right triangle has sides

$$(x, y, z) = \left(-\frac{(t^2 - 6t + 1)(t + 1)^2}{16t^2},\ \frac{(t - 1)^2}{2t},\ \frac{t^4 - 4t^3 + 22t^2 - 4t + 1}{16t^2}\right),$$

and the θ-rational right trapezoid has sides

$$(p, q, h, r) = \left(-\frac{(t + 1)(t^2 - 4t - 1)}{8t},\ \frac{(t + 1)(t^2 + 4t - 1)}{8t^2},\ \frac{(t - 1)^2}{4t},\ \frac{(t^2 + 1)(t - 1)^2}{8t^2}\right).$$

Since v, p, q, h, r are positive rational numbers, $0 < \sin\theta < 1$ and $v < 1$, we obtain the condition

$$0.236067977 < t < 1.$$

Then if $0.236067977 < t < 1$, there are infinitely many rational right triangle and θ-rational right trapezoid pairs with a common area and a common perimeter.

Therefore, there are infinitely many such integral right triangle and θ-integral right trapezoid pairs.

Similarly, we can prove that there are infinitely many such integral right triangle and θ-integral isosceles trapezoid pairs. $\square$

Example 1. If $t = \frac{1}{2}$, we have a right triangle with sides $(x, y, z) = (63, 16, 65)$, and a right trapezoid with sides $(p, q, h, r) = (66, 60, 8, 10)$, which have a common area 504 and a common perimeter 144.

Proof of Theorem 2. As in Theorem 1, we only need to consider the θ-rational isosceles triangle and rational right trapezoid pairs with a common area and a common perimeter. We may set $u = 1$, and v, p, q, h, r to be positive rational numbers, then

$$\begin{cases} 2v(1-v^2) = \dfrac{1}{2}(p+q)h, \\ 4 = p+q+r+h, \\ \dfrac{p-q}{r} = \cos\theta, \\ h = r\sin\theta, \end{cases} \tag{2}$$

where $\sin\theta = 2t/(1+t^2)$ and $\cos\theta = (1-t^2)/(1+t^2)$, with $0 < t < 1$ a rational number.

Eliminating h, p, q in equation (2), we have

$$\frac{t(t+1)^2r^2 - 4t(t^2+1)r - 2v(v-1)(v+1)(t^2+1)^2}{(t^2+1)^2} = 0.$$

It only needs to consider

$$t(t+1)^2r^2 - 4t(t^2+1)r - 2v(v-1)(v+1)(t^2+1)^2 = 0.$$

If this quadratic equation has rational solutions r, then its discriminant $\Delta_2(r)$ should be a rational perfect square. Let us consider the curve

$$\mathcal{C}_2 :\ s^2 = \Delta_2(r) = 2t(t+1)^2v^3 - 2t(t+1)^2v + 4t^2,$$

which is a cubic curve with a rational point $Q_0 = (v, s) = (1, 2t)$. And the curve $\mathcal{C}_2$ is birationally equivalent with the elliptic curve $\mathcal{E}_2$ given by the equation in Weierstrass form

$$\mathcal{E}_2 :\ Y^2 = X^3 - 4t^2(t+1)^4X + 16t^4(t+1)^4.$$

The map $\varphi_2 : \mathcal{C}_2 \ni (v, s) \mapsto (X, Y) \in \mathcal{E}_2$ is given by

$$X = 2vt(t+1)^2,\ Y = 2st(t+1)^2,$$

and the inverse map

$$\varphi_2^{-1} :\ v = \frac{X}{2t(t+1)^2},\ s = \frac{Y}{2t(t+1)^2}.$$

By the map φ_2, we get the point

$$Q = \varphi_2(Q_0) = (2t(t+1)^2,\ 4t^2(t+1)^2),$$

which lies on the elliptic curve $\mathcal{E}_2$. By the group law, we have

$$[2]Q = ((t-1)^2(t+1)^2,\ -(t^4-2t^3-2t^2-2t+1)(t+1)^2).$$

Through the inverse map φ_2^{-1}, we get

$$v = \varphi_2^{-1}(x([2]Q)) = \frac{(t-1)^2}{2t},$$

then

$$\begin{aligned} r =& \frac{(t-1)^2(t^2+1)^2}{2(t+1)^2t^2}, \\ p =& -\frac{t^4-2t^3-2t^2-6t+1}{t(t+1)}, \\ q =& -\frac{t^4-6t^3-2t^2-2t+1}{t^2(t+1)}, \\ h =& \frac{(t-1)^2(t^2+1)}{t(t+1)^2}. \end{aligned}$$

Hence, the rational isosceles triangle has sides

$$\begin{aligned} (x,y,z) = \bigg(& \frac{t^4-4t^3+10t^2-4t+1}{4t^2}, \frac{t^4-4t^3+10t^2-4t+1}{4t^2}, \\ & -\frac{t^4-4t^3-2t^2-4t+1}{4t^2} \bigg), \end{aligned}$$

and the θ-rational right trapezoid has sides

$$\begin{aligned} (p,q,h,r) = \bigg(& -\frac{t^4-2t^3-2t^2-6t+1}{t(t+1)}, -\frac{t^4-6t^3-2t^2-2t+1}{t^2(t+1)}, \\ & \frac{(t-1)^2(t^2+1)}{t(t+1)^2}, \frac{(t-1)^2(t^2+1)^2}{2(t+1)^2t^2} \bigg). \end{aligned}$$

Since v, p, q, h, r are positive rational numbers, $0 < \sin\theta < 1$ and $v < 1$, we obtain the condition

$$0.3137501201 < t < 1.$$

Then if $0.3137501201 < t < 1$, there are infinitely many rational isosceles triangle and θ-rational right trapezoid pairs with a common area and a common perimeter. Therefore, there are infinitely many such integral isosceles triangle and θ-integral right trapezoid pairs.

Similarly, we can prove that there are infinitely many such integral isosceles triangle and θ-integral isosceles trapezoid pairs. □

Example 2. If $t = \frac{1}{2}$, we have an isosceles triangle with sides $(x, y, z) = (153, 153, 270)$ and a right trapezoid with sides $(p, q, h, r) = (258, 228, 40, 50)$, which have a common area 9720 and a common perimeter 576.

Proof of Theorem 3. As in Theorem 1, we only need to investigate the rational Heron triangle and θ-rational right trapezoid pairs with a common area and a common perimeter. We can set $w = 1$, and u, v, p, q, h, r to be positive rational numbers, then

$$\begin{cases} uv(v+1)(u^2-v) = \dfrac{1}{2}(p+q)h, \\ 2u^2(1+v) = p+q+r+h, \\ \dfrac{p-q}{r} = \cos\theta, \\ h = r\sin\theta, \end{cases} \tag{3}$$

where $\sin\theta = 2t/(1+t^2)$ and $\cos\theta = (1-t^2)/(1+t^2)$, with $0 < t < 1$ a rational number.

Eliminating h, p, q in equation (3), we have

$$\frac{t(t+1)^2r^2 - 2tu^2(t^2+1)(v+1)r + uv(v+1)(u^2-v)(t^2+1)^2}{(t^2+1)^2} = 0.$$

It only needs to consider

$$t(t+1)^2r^2 - 2tu^2(t^2+1)(v+1)r + uv(v+1)(u^2-v)(t^2+1)^2 = 0.$$

If this quadratic equation has rational solutions r, then its discriminant $\Delta_3(r)$ should be a rational perfect square. Let us consider the curve

$$\begin{aligned} \mathcal{C}_3:\ s^2 = \Delta_3(r) =&(t+1)^2tuv^3 - (t^2u^2 - tu^3 + 2tu^2 - t^2 + u^2 - 2t - 1)tuv^2 \\ &- u^3(t^2 - 2tu + 2t + 1)tv + t^2u^4, \end{aligned}$$

which is a cubic curve with a rational point $R_0 = (v, s) = (u^2, tu^2(u^2+1))$. And the curve $\mathcal{C}_3$ is birationally equivalent to the elliptic curve $\mathcal{E}_3$ given by the equation in Weierstrass form

$$\mathcal{E}_3:\ Y^2 = X^3 + A_4X + A_6,$$

where

$$\begin{aligned} A_4 =& -27t^2u^2(t^4u^4 - 2t^3u^5 + t^2u^6 + 4t^3u^4 - 4t^2u^5 + t^4u^2 - 4t^3u^3 + 6t^2u^4 \\ &- 2tu^5 + 4t^3u^2 - 8t^2u^3 + 4tu^4 + t^4 + 6t^2u^2 - 4tu^3 + u^4 + 4t^3 + 4tu^2 \\ &+ 6t^2 + u^2 + 4t + 1), \\ A_6 =& -27t^3u^3(t^2u^2 - tu^3 + 2tu^2 + 2t^2 + u^2 + 4t + 2)(2t^4u^4 - 4t^3u^5 + 2t^2u^6 \\ &+ 8t^3u^4 - 8t^2u^5 - t^4u^2 - 8t^3u^3 + 12t^2u^4 - 4tu^5 - 4t^3u^2 - 16t^2u^3 + 8tu^4 \\ &- t^4 - 6t^2u^2 - 8tu^3 + 2u^4 - 4t^3 - 4tu^2 - 6t^2 - u^2 - 4t - 1). \end{aligned}$$

The map $\varphi_3 : \mathcal{C}_3 \ni (v, s) \mapsto (X, Y) \in \mathcal{E}_3$ is given by

$$\begin{aligned} &X = -3tu(t^2u^2 - tu^3 - 3t^2v + 2tu^2 - t^2 - 6tv + u^2 - 2t - 3v - 1), \\ &Y = 27sut(t+1)^2, \end{aligned}$$

and the inverse map φ_3^{-1} :

$$\begin{aligned} v =& \frac{X + 3tu(t^2u^2 - tu^3 + 2tu^2 - t^2 + u^2 - 2t - 1)}{9ut(t+1)^2}, \\ s =& \frac{Y}{27ut(t+1)^2}. \end{aligned}$$

By the map φ_3, we get the point

$$\begin{aligned} R = \varphi_3(R_0) = (&3tu(2t^2u^2 + tu^3 + 4tu^2 + t^2 + 2u^2 + 2t + 1), \\ &27t^2u^3(u^2+1)(t+1)^2), \end{aligned}$$

which lies on the elliptic curve $\mathcal{E}_3$. By the group law, we have

$$\begin{aligned} [2]R = \Big(&\frac{3}{4}u(4t^2u^3 - 4t(t+1)^2u^2 + 3(t+1)^4u - 8t(t+1)^2), \\ &-\frac{27}{8}u^2(t+1)^4(t^2u - 2tu^2 + 2tu - 4t + u)\Big). \end{aligned}$$

Through the inverse map φ_3^{-1}, we get

$$v = \varphi_3^{-1}(x([2]R)) = \frac{t^2u - 4tu^2 + 2tu - 4t + u}{4t},$$

then

$$\begin{aligned} r =& \frac{u(t^2+1)(t^2u + 2tu - 4t + u)}{8t^2}, \\ p =& -\frac{(t+1)(t^2u - 2tu^2 + 2tu - 2u^2 - 4t + u)u}{8t}, \\ q =& \frac{(t+1)(2t^2u^2 - t^2u + 2tu^2 - 2tu + 4t - u)u}{8t^2}, \\ h =& \frac{u(t^2u + 2tu - 4t + u)}{4t}. \end{aligned}$$

To simplify the proof, we set $t = \frac{1}{2}$. Hence, the rational Heron triangle has sides

$$(x, y, z) = \left(\frac{9u(8u^2 - 9u + 8)}{64}, \frac{(9u-8)(u^2+1)}{8}, \frac{145u^2 - 144u + 64}{64}\right),$$

and the θ-rational right trapezoid has sides

$$(p, q, h, r) = \left(\frac{3u(12u^2 - 9u + 8)}{32}, \frac{3u(6u^2 - 9u + 8)}{16}, \frac{u(9u-8)}{8}, \frac{5u(9u-8)}{32}\right).$$

Since u, v, p, q, h, r are positive rational numbers, $0 < \sin\theta < 1$ and $u^2 > v$, we obtain the condition

$$u > \frac{8}{9}.$$

Then if $u > \frac{8}{9}$, there are infinitely many rational Heron triangles and θ-rational right trapezoid pairs with a common area and a common perimeter. Therefore, there are infinitely many such Heron triangles and θ-integral right trapezoid pairs.

Similarly, we can prove that there are infinitely many such Heron triangles and θ-integral isosceles trapezoid pairs. $\square$

Example 3. If $u = 2$, we have a Heron triangle with sides $(x, y, z) = (99, 100, 89)$ and a right trapezoid with sides $(p, q, h, r) = (114, 84, 40, 50)$, which have a common area 3960 and a common perimeter 288.

3. Some related questions

We have studied the integral triangles and θ-integral trapezoids pairs with a common area and a common perimeter. Noting that isosceles trapezoid is a cyclic quadrilateral, so it is also interesting to consider the following three questions.

Question 1. Are there infinitely many integral right triangle and θ-integral cyclic (resp. rational) quadrilateral pairs with a common area and a common perimeter?

Question 2. Are there infinitely many integral isosceles triangle and θ-integral cyclic (resp. rational) quadrilateral pairs with a common area and a common perimeter?

Question 3. Are there infinitely many Heron triangle and θ-integral cyclic (resp. rational) quadrilateral pairs with a common area and a common perimeter?

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