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Pairs of Congruent-Like Quadrilaterals that are not Congruent

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Abstract. In this article we will give a negative answer to a question referred to a congruence theorem for quadrilaterals that have been recently introduced in [5, section 4]. Precisely we will show that there exist pairs of quadrilaterals having 8 pieces (four sides and four angles) pairwise congruent, but that are not congruent. Computations and the use of geometric design by dynamical software will have a crucial role in the proof.

1. Introduction

One of the main topic in Mathematical Geometric Foundations is the concept of 'congruence' between polygons. Two convex polygons are said to be *congruent* if there is a one-to-one correspondence between their vertices such that consecutive vertices correspond to consecutive vertices, and all pairs of correspondent sides and all pairs of correspondent angles are congruent. Formally, we can say that two polygons are congruent if, and only if, one can be transformed into the other by an isometry.

Usually congruence theorems for triangles, for quadrilaterals and in general for polygons, are stated using sides and angles of pairs of such figures (see [4], [6], [11]); but for polygons certain remarks are required (see [4, Lesson 11] and [6, Chapter 8] for the definitions and the development of the basic properties related to polygons).

The general study of congruence between two polygons seems too hard, and different cases such as convex (see [4, Definition 9.7]), non convex or twisted polygons should be considered.

Extending a definition introduced in [5] we will say that two polygons P and P' are *congruent-like*, if there is a bijection between the sides of P and P', and a bijection between the angles of P and P', such that each side of P is congruent to a corresponding side of P' and each angle of P is congruent to a corresponding angle of P' (note that the two bijections are 'a priori' independent).

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Clearly, two congruent n-gons are congruent-like. It is easy to check that the converse holds for pairs of triangles; but, as remarked in [5], it is not always true if $n > 4$ (see Figures 1 and 2).

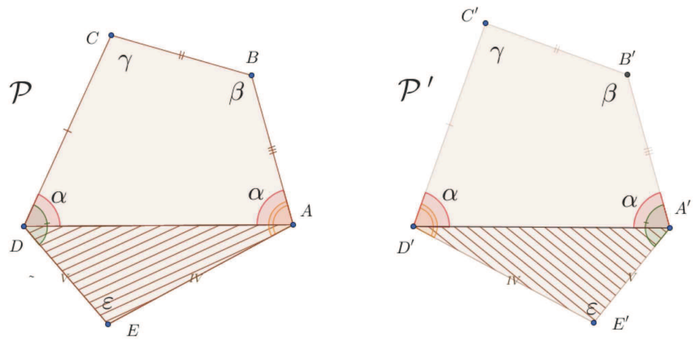

Figure 1. Starting from two congruent quadrilaterals (A, B, C, D) and (A', B', C', D') with two congruent consecutive angles of magnitude α, we may easily construct pairs of congruent-like polygons that are not congruent.

Following [5], in this paper we will denote by $P = (A_0, \dots, A_{n-1})$ the polygon whose consecutive vertices are $A_0, \dots, A_{n-1}$ (usually numbered counterclockwise), by a_i the length of the side A_iA_{i+1} (the index i is read mod n). The inner angle at the vertex A_i as well as its measure will be denoted by $\hat{A}_i$, $A_{i-1}\hat{A}_iA_{i+1}$ or simply by α_i. Moreover, we will denote by:

$$a_i \to a_{i+1} \to \dots \to a_{i+4} \to a_{i+5} \to a_i, \quad \text{or}$$

$$A_iA_{i+1} \to A_{i+1}A_{i+2} \to \dots \to A_{i+n-2}A_{i+n-1} \to A_{i+n-1}A_i \to A_iA_{i+1},$$

the ordered sequence of the 'sides' of P, starting from A_iA_{i+1} (see Figure 2).

We will say that two congruent-like polygons P and P' are *ordered congruent-like polygons*, if whichever ordered sequence of consecutive sides of P is equal to an ordered sequence of sides of P'. In [5, Theorem 3.5] it has been recently showed that convex ordered congruent-like quadrilaterals are indeed congruent (a complete list of other classical congruence theorems for quadrilaterals can be found in [4] and [6] for instance).

Theorem 1. *Let Q and Q' be ordered congruent-like convex quadrilaterals. Then Q and Q' are congruent.*

This result cannot be extended to hexagons (and any other n-gons, for $n \geq 6$), as Figure 2 shows, and it is still an open question for (ordered) pentagons.

Another natural question that arises is:

P) *Can we remove the hypothesis that Q and Q' are ordered?*

The aim of this article is to investigate this question. As we will see examples of (non-ordered) convex congruent-like quadrilaterals Q and Q' that are not congruent exist.

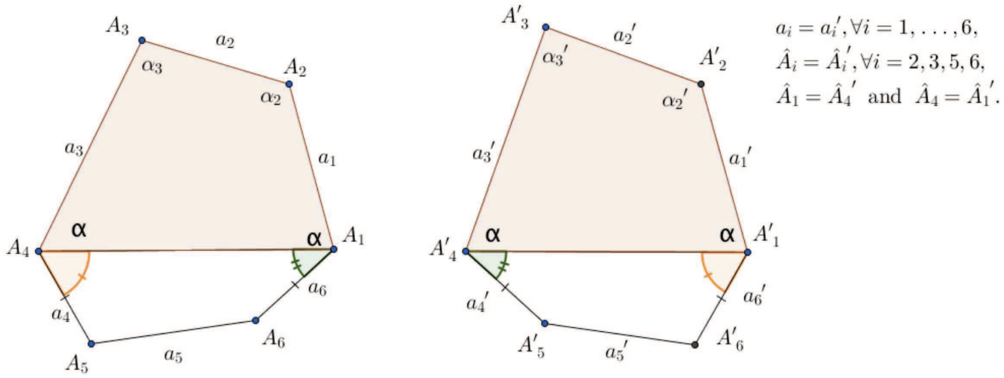

Figure 2. Starting from a quadrilateral (A, B, C, D) with two congruent consecutive angles, say α, we may easily construct pairs of ordered congruent-like n-gons that are not congruent, for every $n > 5$. Here any sequence $a_i \to a_{i+1} \to \ldots \to a_{i+4} \to a_{i+5} \to a_i$ of sides of the hexagon on the left is equal to $a_i' \to a_{i+1}' \to \ldots \to a_{i+4}' \to a_{i+5}' \to a_i'$ referred to the hexagon on the right.

As we will see the construction of such examples will be not so elementary: software for geometric designs, computation and combined arguments of synthetic geometry and analysis will be used. Here we only prove the existence, and on the other hand we will also provide for a method to obtain good approximations of such pairs of quadrilaterals that are not congruent with all sides and angles pairwise congruent (see Figure 8).

In section 2, we will start to define families $\mathcal{F}_{m,\alpha}$ of pairs of quadrilaterals $Q_{d,a}$ and $Q'_{d,a}$ (depending on two parameters, namely d and a that are also two consecutive sides of the $Q_{d,a}$), which have five elements pairwise congruent (three sides and two angles), and related 'functions' which describes the relations among sides and angles of $Q_{d,a}$ and $Q'_{d,a}$. In section 3, we will use dynamic software for restricting the study of 'functions' associated to pairs of quadrilaterals to smaller intervals.

In section 4 we will highlight the properties of those 'functions' introduced in section 2. In section 5 we will prove that convex congruent-like quadrilaterals Q and Q' that are not congruent exist!

The paper is suitable for a large audience of readers who are referred to [9], [10], [7] and reference therein, for deeper investigations on quadrilaterals and related topics.

The measure of angles that occur in this paper are expressed in degrees (instead of radians). This will make easier the reading of the figures for those angles close to the right one (see Lemma 4 for example); moreover in the computations, this will make more apparent the difference between the measure of angles and the measure of sides, as in Lemmas 9 and 10).

2. Families of pairs of quadrilaterals having three sides and two angles pairwise congruent

We begin presenting a technique for building pairs of quadrilaterals with at least five elements pairwise congruent, precisely three sides and two angles. Let's proceed by steps:

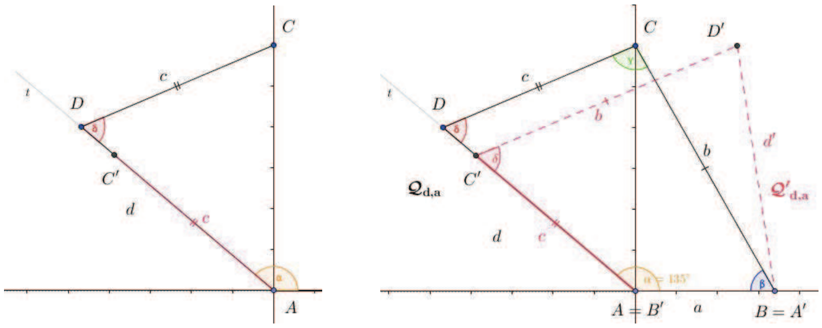

Figure 3. Two steps for constructing pairs of quadrilaterals with 5 pieces congruent. Note that $d = AD$, and $c = CD = AC'$.

1) Fix a point A at the origin of a coordinate system, and choose a point C on the (positive) y-axis at a distance say m from A. After, consider an obtuse angle α whose sides are the (positive) x-axis and t - which is a line lying in the second quadrant (see Figure 3 on the left). For every positive integer $d \in I = [m/\sqrt{2}, m\sqrt{2}]$, on the side t we consider a point D such that $d = AD$. Clearly $AD \geq DC$, and the smaller sides of the triangle (A, C, D) are $d = DA$ and $c = CD$. Now, let C' be the point lying on the line t such that the measure of $B'C'$ is c (see Figure 3).

2) Let now, a be a positive real number, and let B lie on the positive part of the x-axis, such that $AB = a$ (look at Figure 3 on the right). In this way two quadrilaterals are determined: $Q = Q_{d,a} = Q(A, B, C, D)$, where B has coordinates $(0, a)$; and $Q' = Q'_{d,a} = Q'(A', B', C', D')$, where $A' = B, B' = A$, and the vertex D', is obtained taking on the parallel line to DC trough C' a segment of length $b = BC$.

The pair $(Q_{d,a}, Q'_{d,a})$ obtained as above will be called m, α*-pair, determined by* (d, a).

In this way for every possible choice of m and α, we obtain the (m, α)*-family*

$$\mathcal{F}_{m,\alpha} = \{(Q = Q_{d,a}, Q' = Q'_{d,a})\}_{(d,a)\in I\times\mathbb{R}^+}$$

that is the collection of all (m, α)*-pairs of quadrilaterals* determined by all the possible choice of (d, a) in $I \times \mathbb{R}^+$. By construction, each pair o quadrilaterals have three sides and two angle pairwise congruent.

As our aim is just to find an example, in order to simplify the computations and to avoid formal complications, we will assume that $m = 3$, $\alpha = 135°$, thus we will focus our attention on the family of pairs $\mathcal{F}_{3,135°}$.

By construction, we have that the "pieces" of $Q = Q_{d,a}$ e $Q' = Q'_{d,a}$ can be considered as functions depending on $d \in [3/\sqrt{2}, 3\sqrt{2}]$ and a positive real number a. We will call these functions as *associated functions to the family* $\mathcal{F}_{3,135^\circ}$. Elementary geometry shows that:

$$c(d) = \sqrt{3^2 + d^2 - 2d\,3\ \cos 45^\circ}. \tag{1}$$

Arguing on the triangle (C, D, A) we have $3^2 = c(d)^2 + d^2 - 2dc(d)\ \cos(\delta(d))$

$$\delta(d) = \arccos\left(\frac{c(d)^2 + d^2 - 3^2}{2d\ c(d)}\right). \tag{2}$$

$$b(a) = \sqrt{a^2 + 3^2}. \tag{3}$$

Let now $u(d, a)$ and $v(d, a)$ be the coordinates of D'. Easy geometric considerations (see Figure 4) yield

$$\begin{aligned} u(d,a) =&\ -c(d)\sqrt{2}/2 + b(a)\ \cos(\alpha + \delta - 180^\circ);\text{ and} \\ v(d,a) =&\ c(d)\sqrt{2}/2 + b(a)\ \sin(\alpha + \delta - 180^\circ); \qquad (4) \\ d'(d,a) =&\ \sqrt{(u(d,a) - a)^2 + v(d,a)^2}. \qquad (5) \end{aligned}$$

Moreover, arguing on the triangle (A, B, D'), we have $AD'^2 = a^2 + d'^2 - 2ad' \cos(\alpha')$, so that:

$$\begin{aligned} \alpha'(d,a) =&\ D'\hat{A}'B' = \arccos\left(\frac{a^2 + d'(d,a)^2 - u(d,a)^2 - v(d,a)^2}{2a\ d'(d,a)}\right), \qquad (6) \\ \beta(d,a) =&\ \arctan(h/a), \qquad (7) \\ \gamma(d,a) =&\ 360^\circ - 135^\circ - \beta(d,a) - \delta(d). \qquad (8) \end{aligned}$$

Here we highlight that each of the above function associated to the family $\mathcal{F}_{3,135^\circ}$ is continuous.

We conclude this section with some remarks:

*R*emark 1. In the above construction we may think to fix d, and move a as variable. In this case, we may note that:

i) If a is close to 0, then $\gamma(d, a)$ is acute and $\alpha'(d, a) = D'\hat{B}A$ is obtuse. Indeed, as $b > c$, then one can check that D' lies in the I quadrant.
ii) If a is enough large, then $\gamma(d, a)$ is obtuse. Precisely, if we put $a = m = CA$, then $\beta = 45^\circ$, $B\hat{C}A = 45^\circ$. Moreover, as $c < d$ we have $D\hat{C}A > 45^\circ$. Therefore, $\gamma(d, a)$ is obtuse.

We also remark that when we fix d, as we will see, it is not trivial to establish the monotony of the functions $\alpha'(d, a)$, $\delta'(d, a)$, $\gamma(d, a)$, and $d'(d, a)$. On the other hand an investigation by dynamic software tools can give useful suggestions in this direction.

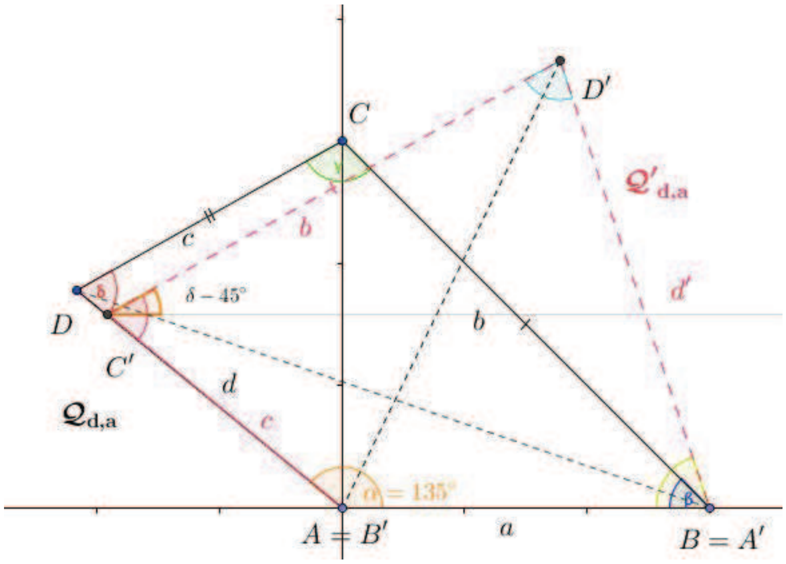

Figure 4. Geometric considerations to determine analytically the 'associated functions' to the family $\mathcal{F}_{3,135°}$.

3. The use of software for finding the approximation of the solution

In this section, we will study some properties related to the associated functions. As first step, we fix d, and in order to select pairs of quadrilaterals belonging to $\mathcal{F}_{3,135°}$ with all the angles pairwise congruent, we will investigate how much is the difference between the angles $\gamma(d,a)$ and $\alpha'(d,a)$ of the pairs $(Q_{d,a}, Q'_{d,a})$. Therefore, we draw the graphics of the following functions:

$$G_d : a \in \mathbb{R}^+ \to \alpha'(d,a) - \gamma(d,a) \tag{9}$$

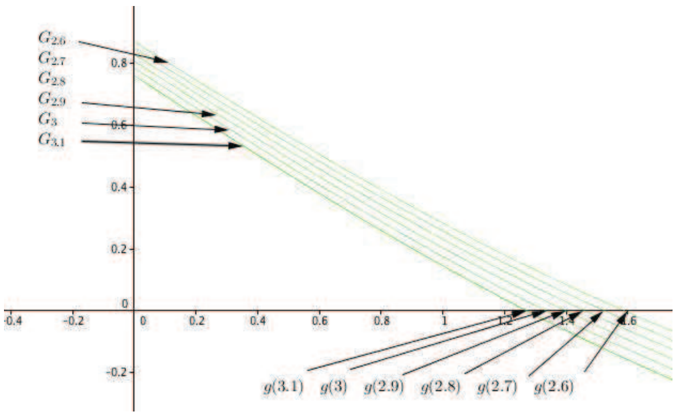

Figure 5. Some graphics related to G_d.

Looking at the Figure 5, we may think that for every $d = 2.6, \ldots 3.1$, that G_d has a zero in $g(d)$ (we also note that $g(d)$ seems univocally determined), and therefore the angles $\gamma(d, g(d))$ and $\alpha'(d, g(d))$ of $(Q_{d,g(d)}, Q'_{d,g(d)})$ coincides. For these reasons (see also section 4) we will call g as *angular synchronization function*.

In this way we can 'determine' many pairs of quadrilaterals - $(Q_{d,g(d)}, Q'_{d,g(d)})$ - with seven elements (three sides and four angles) pairwise congruent.

Now, the problem is:
Is there among them one with all pieces pairwise congruent? or equivalently
Does there exist $\overline{d}$ such that $\overline{d} = d'(\overline{d}, g(\overline{d}))$? In other term:
P): *Can we determine $\overline{d}$ such that $(Q_{\overline{d},g(\overline{d})}, Q'_{\overline{d},(\overline{d})})$ are congruent-like?*

In order to reduce the interval in which to find a such $\overline{d}$, we combine the functions $\{G_d\}$ with the functions that describes the difference between $d'(d, a)$ and d, whenever d is fixed:

$$H_d(a) : a \in \mathbb{R} \to d'(d, a) - d. \tag{10}$$

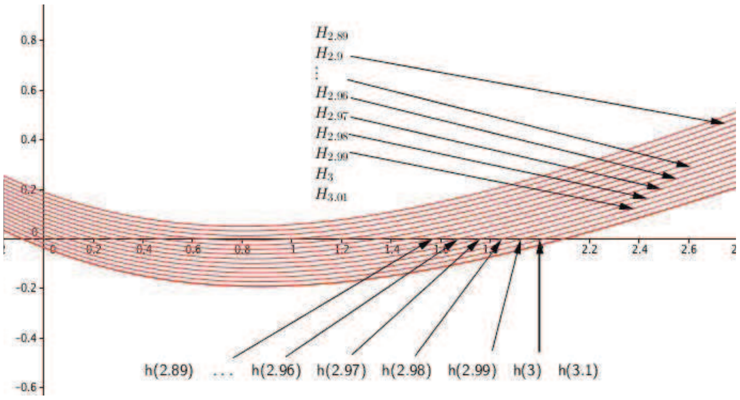

Figure 6. Some graphics related to H_d.

Clearly both G_d and H_d are continuous functions of the variable a.

We note that, if we consider small values of d, for example if we put $d = 2.9 \in [3/\sqrt{2}, 3\sqrt{2}]$, then $H_d(a) > 0, \quad \forall a \in (0, 2)$. It follows that $\forall a \in (0, 2)$ the side $A'D' = d'(2.9, a)$ of $Q'_{2.9,a}$ is greater than the side $AD = d$ of $Q_{2.9,a}$.

Let $h(d)$ such that $H_d(h(d)) = 0$, then $d = d'(d, h(d))$; but here it is not to say that the quadrilaterals $(Q_{d,h(d)}, Q'_{d,h(d)})$ are congruent-like. On the other hand if we find $\overline{d}$ such that $g(\overline{d}) = h(\overline{d})$, then $G_{\overline{d}}(g(\overline{d})) = H_{\overline{d}}(h(\overline{d})) = 0$, $\overline{d} = d'(\overline{d}, g(\overline{d})$, so that the quadrilaterals $(Q_{\overline{d},g(\overline{d})}, Q'_{\overline{d},(\overline{d})})$ are congruent like.

Remark 2. By comparing the graphics of $\{G_d\}$ and $\{H_d\}$ (see Figure 7) we see that a possible value $\overline{d}$ such that $g(\overline{d}) = h(\overline{d})$ lies in $U = (2.93, 2.97)$. Looking at the Figures 7 and 5, we also note that if d runs in U, then $g(d) < 2$. For these

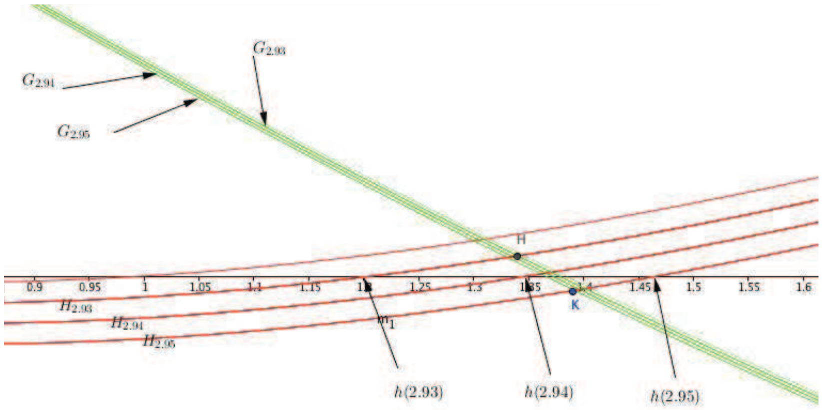

Figure 7. The intersection K between $G_{2.95}$ and $H_{2.95}$ and the intersection H between $G_{2.93}$ and $H_{2.93}$ are farer from the x-axis than the intersection between $G_{2.94}$ and $H_{2.94}$. Therefore, $\overline{d}$ is closer to 2.94 and $g(\overline{d})$ is about 1.37.

reasons we will restrict our investigation to pairs $\{(Q_{d,a}, Q'_{d,a})\}_{(d,a)\in U\times V}$, where $V := (0, 2)$.

Remark 3. If we refine the above graphics by using reduced scale, we may determine more and more precise values for $\overline{d}$ (see Figure 8 on the left), which allow to detect better approximations of other pairs of quadrilaterals we are looking for (see for example Figure 8 on the right).

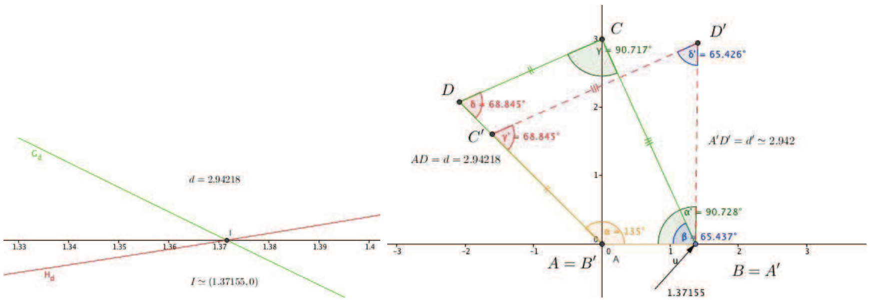

Figure 8. Using a large scale, we can establish that $\overline{d}$ is about 2.94218 and it follows that $g(\overline{d}) \simeq 1.37155$. Note that $\mathcal{Q} = (A, B, C, D)$ and $\mathcal{Q}' = (A', B', C', D')$ are not congruent, but have 5 pieces (three sides and two angles) pairwise congruent by construction. Moreover it is also $AD \simeq A'D'$, $\gamma \simeq \alpha'$ and $\beta \simeq \delta'$.

Even if these 'qualitative' consideration do not prove the existence of $\overline{d}$, they will be crucial in order to study the associated functions in suitable intervals. In the next sections, in order to prove the main theorem, we will formalize analytically the behavior of the associated functions.

4. The analytic study of the associated functions to pairs of $\mathcal{F}_{3,135^\circ}$

In this section we will study some properties related to the associated functions to pairs of $\mathcal{F}_{3,135^\circ}$. In particular, we will prove that $\forall d \in [2.93, 2.97]$ the function

$$G_d : (0, 2] \to \alpha'(d, a) - \gamma(d, a)$$

strictly increases, and has a zero.

This will permit to define analytically the *angular synchronization* function (that we have already mentioned at the beginning of section 3) as:

$$g : d \in (2.93, 2.97) \to g(d) \in (0, 2),$$

where $g(d)$ is the unique real number such that $G_d(g(d)) = 0$.

The proof of the following Lemma is immediate.

Lemma 2. *Let (A, L, M) a triangle. On the line trough A and L let consider a segment $AS > AL$ and then consider an angle $\sigma > \lambda$ such that $\sigma + \alpha < 180°$ (see Figure 9). Then the side s of σ intersects the line trough A and M in a point, say N, such that $LM < SN$ and $MA < NA$.*

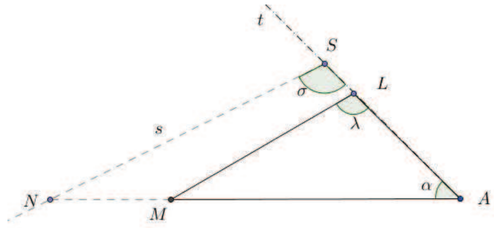

Figure 9. Inequalities between triangles.

The following two Lemmas give universal - that is $\forall d \in [2.93, 2.97]$ - inequalities referred to $\gamma(d, a), \alpha'(d, a), \delta'(d, a)$ when $a = 2$. We highlight that the proofs will use both analytic and syntethic arguments.

Lemma 3. *Let $a = 2$. Then $\forall d \in [2.93, 2.97]$ we have $\alpha'(d, 2) > \delta'(d, 2)$.*

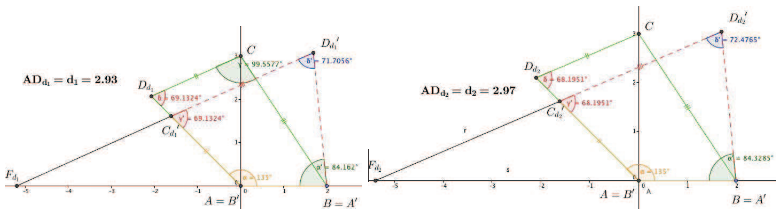

Figure 10. Note that the vertices F, C' e D' and D depend on d. Two extreme cases are related to pair ($\mathcal{Q}_{2.93,2}$, $\mathcal{Q}'_{2.93,2}$) and pair ($\mathcal{Q}_{2.97,2}$, $\mathcal{Q}'_{2.97,2}$).

Proof. 1) If $d_1 = 2.93$ (see Figure 10 on the left), then considering $\mathcal{Q}'_{d_1,2}$ we have: $3.925 < F_{d_1}C'_{d_1}$ and $5.188 < F_{d_1}A$, and in particular

$$F_{d_1}B = F_{d_1}A + a < 5.2 + 2 < 3.925 + \sqrt{13} \leq F_{d_1}C'_{d_1} + b = F_{d_1}D'_{d_1}.$$

2) If $d_2 = 2.97$ (see Figure 10 on the right), then considering $\mathcal{Q}'_{d_2,2}$ we have: $4.101 < F_{d_2}C'_{d_2} \simeq 4.102 < 4.103$ and $F_{d_2}A \simeq 5.386$, so that

$$F_{d_2}B = F_{d_2}A + a < 5.387 + 2 < 4.101 + \sqrt{13} \leq F_{d_2}C'_{d_2} + b = F_{d_2}D'_{d_2}.$$

In both the cases ($i = 1, 2$), looking at the triangle (F_{d_i}, D'_{d_i}, A'), the side $F_{d_i}D'_{d_i}$ is greater than $F_{d_i}B$, so that the same relation holds for the opposite angles, that is $\alpha'(d_i, 2) > \delta'(d_i, 2)$.

Let now $d \in (2.93, 2.97)$. Clearly $\delta(d_1, 2) > \delta(d, 2) > \delta(2.97, 2)$, and $c(2.93) < c(d, 2) < c(2.97)$. Then, if we consider the triangles (C'_{d_1}, A, F_{d_1}), (C'_d, A, F_d) and (C'_{d_2}, A, F_{d_2}), by Lemma 2 we have the following inequalities:

$$3.925 < F_{d_1}C'_{d_1} < F_dC'_d < F_{d_2}C'_{d_2} \simeq 4.102 < 4.103 \quad \text{and}$$
$$5.188 < F_{d_1}A \simeq 5.1885 < F_dA < F_{d_2}A \simeq 5.386 < 5.387.$$

It follows that $F_dA_d - F_dC'_d < 5.387 - 3.925 = 1.462 < b - a\sqrt{9+4} - 2 = b - a$.

Finally, looking at the triangle (F_d, D'_d, B) we have $F_dB = F_dA_d + a < F_dC'_d + b = F_dD'_d$, so that $\alpha'(d, 2) > \delta'(d, 2)$. □

Lemma 4. *Let* $a = 2$. *Then* $\forall d \in [2.93, 2.97]$ *the angle* $\gamma(d, 2)$ *is obtuse and* $\alpha'(d, 2)$ *is acute.*

Proof. Clearly, whenever we fix $a \in (0, 2]$, the function $\gamma^a : d \in [2.93, 2.97] \to \gamma(d, a)$ increases. On the other hand, a computation yields $99.55° < \gamma(2.93, 2)$, so that

$$\forall d \in (2.93, 2.97) \quad \text{we have} \quad 99.55° < \gamma(2.93, 2) < \gamma(d, 2),$$

and the first part of the statement holds.

To show that $\alpha'(d, 2)$ is acute we will bound all the associated functions that occur to determine it. Using Equation 6 we have:

$$\alpha'(d, 2) = D'\hat{A}'B' = \arccos\left(\frac{2^2 + d'(d, 2)^2 - u(d, 2)^2 - v(d, 2)^2}{2 \cdot 2\, d'(d, 2)}\right). \tag{11}$$

We make the following preliminary remarks:

1) The associated function $c(d)$ (see Equation 1) increases -This can be checked by synthetic geometric way-.

2) By hypothesis $b = \sqrt{AC^2 + 2^2} = \sqrt{13}$ is fixed.

3)The following inequalities can be obtained by a direct computation by using Equations 1 – 8.

$$\begin{array}{ccc} & d=2.97 & \\ 68.195^\circ < & \delta(2.97,2) < & 68.196^\circ \\ 84.328^\circ < & \alpha'(2.97,2) < & 84.329^\circ \\ 100.495^\circ < & \gamma(2.97,2) < & 100.496^\circ \end{array} \quad \text{and} \quad \begin{array}{ccc} & d=2.93 & \\ 69.132^\circ < & \delta(2.93,2) < & 69.133^\circ \\ 84.161^\circ < & \alpha'(2.93,2) < & 84.162^\circ \\ 99.557^\circ < & \gamma(2.93,2) < & 99.558^\circ. \end{array} \tag{12}$$

In particular $\alpha'(d,2)$ is acute when $d = 2.93$ or $d = 2.97$.

Let now $2.93 < d < 2.97$. In the next four steps we will bound the functions $c(d), u(d,2), v(d,2)$ and $d'(d,2)$:

- We have just highlighted that $c(d)$ is an increasing function in $(2.93, 2.97)$, thus by Equation 1 we have the following bound in $(2.93, 2.97)$:

$$\begin{aligned} &\sqrt{3^2 + 2.93^2 - 2 \cdot 2.93 \cdot \sqrt{2}/2} \le c(d) \le \sqrt{3^2 + 2.97^2 - 2 \cdot 2.97 \cdot \sqrt{2}/2} \\ \iff &\sqrt{5.15396} = 2.27 \le c(d) \le \sqrt{5.2202} = 2.2848. \end{aligned} \tag{13}$$

- To bound $u(d,2)$ in $(2.93, 2.97)$, we recall that (see Equation 4)

$$u(d,a) = -c(d)\sqrt{2}/2 + b(a)\ \cos(\delta(d) - 45^\circ).$$

  As $(2.93, 2.97) \subset (3/\sqrt{2}, 3)$, we have that $\delta(d)$ lies in $(45^\circ, 90^\circ)$, $\forall d \in (2.93, 2.97)$ (see Figure 4). In particular $0^\circ < \delta(d) - 45^\circ < 45^\circ$.

  We have noted that $c(d)$ is an increasing function in $(2.93, 2.97)$, while the cosine function is decreasing in $(0^\circ, 90^\circ)$. Thus, by Relations (12) we have the following bounds for $u(d,2)$:

$$\begin{aligned} &- c(2.97)\sqrt{2}/2 + \sqrt{13}\ \cos(69.133^\circ - 45^\circ) \le u(d,2) \\ &\le -c(2.93)\sqrt{2}/2 + \sqrt{13}\ \cos(68.195^\circ - 45^\circ) \qquad (14) \\ \iff &- 2.2848\sqrt{2}/2 + \sqrt{13}\ \cos(24.133^\circ) \le u(d,2) \\ &\le -2.27\sqrt{2}/2 + \sqrt{13}\ \cos(23.195^\circ) \\ \iff &1.6748 \le u(d,2) \le 1.7089. \end{aligned}$$

- Similarly, taking into account that the sine function is increasing in $(0^\circ, 90^\circ)$ we can bound $v(d,2)$. Indeed by Equation 4 we have

$$v(d,2) = c(d)\sqrt{2}/2 + b(a)\ \sin(\alpha + \delta(d) - \pi),$$

  so that

$$\begin{aligned} &c(2.93)\sqrt{2}/2 + \sqrt{13}\ \sin(23.195^\circ) \le v(d,2) \\ &\quad \le c(2.97)\sqrt{2}/2 + \sqrt{13}\ \sin(24.133^\circ) \qquad (15) \\ \iff &3.0252 \le v(d,2) \le 3.0897. \qquad (16) \end{aligned}$$

- To bound $d'(d,2)$ in $(2.93, 2.97)$ we can use Equation 5:

$$d'(d,2) = \sqrt{((u(d,2)-2)^2 + v(d,2)^2}. \tag{17}$$

By Equation 14, $1.6748 \leq u(d,2) \leq 1.7089$, then $(1.7089-2)^2 < (u(d,2)-2)^2 < (1.6748-2)^2$, and by relation 15 we have the following bound:

$$3.03917 < \sqrt{9.23658} < d'(d,2) < \sqrt{9.65201} < 3.1067.$$

Now we show that $\alpha'(d,2)$ is acute. Looking at relation 11, we note that the value "$4\ d'(d,2)''$ is always a positive real number, so that it suffices to show that the numerator inside the argument of the trigonometric function $\arccos$ is positive. Thus we have to show that

$$2^2 + d'(d,2)^2 \geq u(d,2)^2 + v(d,2)^2.$$

By relations 14, 15, and 17 we have the following inequalities $\forall d \in (2.93, 2.97)$:

$$\begin{aligned} u(d,2)^2 + v(d,2)^2 &< 1.7089^2 + 3.0897^2 = 12.4665 \\ &\leq 13.2365 \leq 4^2 + 3.03917^2 \\ &\leq 2^2 + d'(d,2)^2. \end{aligned}$$

The lemma is proved. □

We also need the following elementary properties, that in particular holds for rectangular triangles.

Lemma 5. *Let (A,B,C) a triangle, and let (A,B_1,C) a triangle as in Figure 11. Then $CB - CB_1 < BB_1$.*

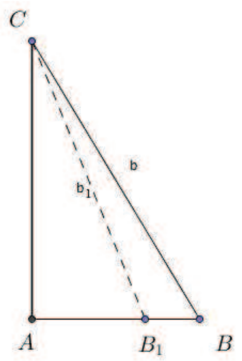

Figure 11. If $BA > B_1A$ and $BA - B_1A = \varepsilon$, then the hypotenuse of the smaller triangle AB_1C is shorter than the hypotenuse of the triangle ABC; moreover their difference $b - b_1$ is less than ε.

Proof. It is enough to apply the triangle inequality to the triangle (B, B_1, C). □

Now we are in a position to show that $\forall d \in (2.93, 2.97)$ the function

$$G_d|(0,2] : a \in (0,2] \to \alpha'(d,a) - \gamma(d,a)$$

strictly decreases (see also Definition 9 and Figure 6). Indeed the next Proposition shows that when we fix $d \in (2.93, 2.97)$, if a decreases, then $\alpha'(d, a)$ increases; while $\gamma(d, a)$ and $\delta'(d, a)$ decreases.

Proposition 6. *Let $d \in [2.93, 2.97]$, and let AB_1 and AB be segments whose length respectively a_1 and a, are such that $a_1 < a \le 2$. Then the following relations referred to angles of $Q'_{d,AB} = (A', B', C', D')$, $Q'_{d,AB_1} = (A', B'_1, C', D'_1)$, $Q_{d,AB} = (A, B, C, D)$ and $Q_{d,AB_1} = (A, B_1, C, D)$ hold (see Figure 12).*

(i) $\gamma = \gamma(d, a) = D\hat{C}B > \gamma_1 = \gamma(d, a_1) = D\hat{C}B_1$.
(ii) $\alpha' = \alpha'(d, a) = D'\hat{A}'B' < \alpha'_1 = \alpha'(d, a_1) = D'_1\hat{B}_1B'$.
(iii) $\delta' = \delta'(d, a) > \delta'_1 = \delta'(d, a_1)$.

In particular $G_d|(0, 2] : a \in (0, 2] \to \alpha'(d, a) - \gamma(d, a)$ strictly decreases.

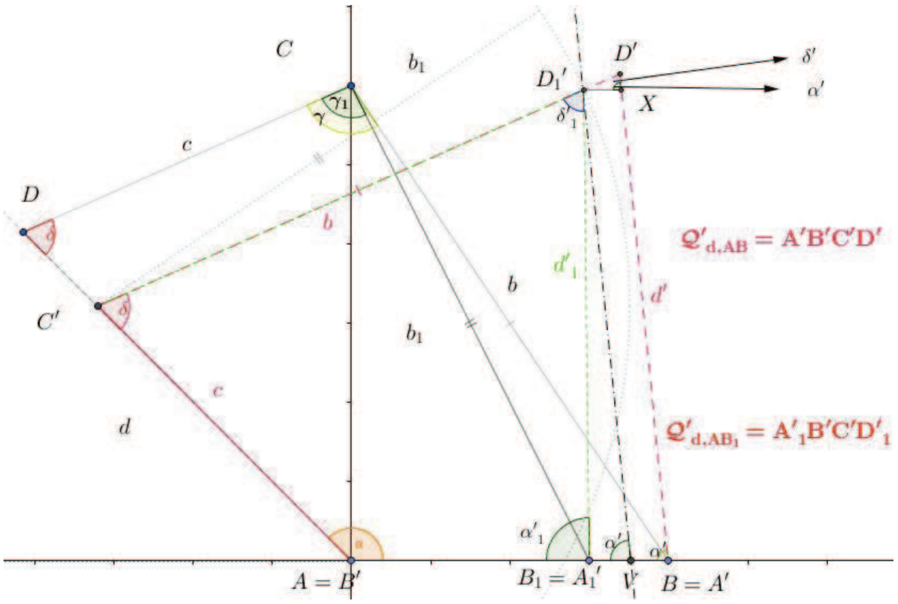

Figure 12. The monotony of the functions $\gamma(d, a)$, $\alpha'(d, a)$ and $\delta'(d, a)$, where d is fixed in $[2.93, 2.97]$.

Proof. (i) This condition follows by elementary geometry. If $a < a_1$ then $\gamma > \gamma_1$ (look at the Figure 12).

(ii) In order to show this condition we suppose first that $a = AB = 2$. By Lemma 5 the length $b - b_1$ of $D'D'_1$ is less than $A'A'_1$. Put V the intersection between the parallel line to $A'D'$ trough D'_1 and AB, and put X the intersection between $D'A'$ with the parallel line to AB trough D'_1. It turns out that (V, A', X, D'_1) is a parallelogram.

Note that V lies on the left of A'. By Lemma 3 we have $\alpha'(d, 2) > \delta'(d, 2) = C'\hat{D}'A'$, therefore looking at the triangle (D', D'_1, X) we have that $D'_1X = BV$ is less than D'_1D'.

By contradiction suppose that V doesn't belong to $A'A'_1$ (that is 'V lies on the left of A'_1') we have $A'V = D'_1X > A'A'_1 > D'D'_1$, and this is a contradiction.

Therefore V belongs to the segment B_1B, and the external angle $\alpha'_1 = D'_1\hat{B}_1B'$ is larger than $\alpha' = D'_1VB_1$.

Let now $AB < 2$, and put $z = 2$ e $z_1 := AB$. The first part of the proof shows in particular that $\alpha'(d, z_1) > \alpha'(d, z) > \delta'(d, z) > \delta'(d, z_1)$. This consideration yields that for every $a \leq 2$ the inequality $\alpha'(d, a) > \delta'(d, a)$- which extends Lemma 3 to every $a \in (0, 2]$ - holds. Then it is possible to run over the argument that we have used above. This complete the proof of (ii).

To show the condition (iii), it is enough to remember that by construction the quadrilaterals $Q'_{d,AB}$ and Q'_{d,AB_1} have a common angle of magnitude $135°$, and another one of magnitude $\delta(d, a)$. Thus the condition is proved applying (ii).

□

Remark 4. We highlight that the monotony of G_d is not guaranteed all over $\mathbb{R}^+$.

Proposition 7. *Let* $d \in (2.93, 2.97)$. *Then there exists (exactly one) real number* $g(d) \in (0, 2)$ *such that the angle* $\gamma(d, g(d))$ *of* $Q_{d,g(d)}$ *and the angle* $\alpha'(d, g(d))$ *di* $Q'_{d,g(d)}$ *coincides. In particular* $G_d(g(d)) = 0$*, for every* $d \in (2.93, 2.97)$.

Proof. For every $d \in (2.93, 2.97)$, we have seen (Remark 1) that if a is enough small, then $\gamma(d, a)$ is less than $\alpha'(d, a)$; moreover we have proved (Lemma 4) that if $a = 2$ then $\gamma(d, a) > 90° > \alpha'(d, 2)$.

It follows that for every $d \in (2.93, 2.97)$, the function

$$G_d|_{(0,2]} : a \in (0, 2] \to \alpha'(d, 2) - \gamma(d, a) \in \mathbb{R},$$

assumes both positive and negative values, so that, it has a zero in $(0, 2)$. Moreover by Proposition 6 G_d has only one zero. □

Remark 5. In virtue of Proposition 7 we can define

$$g : d \in (2.93, 2.97) \to g(d) \in (0, 2), \quad \text{such that } G_d(g(d)) = 0.$$

According to Section 3, this function shows the exact value $a = g(d)$, for which we have the 'synchronization' of all the angles of pairs of quadrilaterals of the type $(\mathcal{Q}_{d,a}, \mathcal{Q}'_{d,a})$ that have the same first parameter d. In fact, for every $d \in (2.93, 2.97)$ we have that $\mathcal{Q}_{d,g(d)}$ and $\mathcal{Q}'_{d,g(d)}$ have all the angles pairwise congruent. We may call such pairs *almost congruent quadrilateral*, extending a terminology used for triangles (see [3][1, section 2] for definition and recent results connected them).

In order to show that g is a continuous function, we point out some considerations.

Let f be the following function of two variables defined in an open interval of $\mathbb{R}^2$. Precisely:

$$f : (d, a) \in (2.93, 2.97) \times (0, 2) \to G_d(a) = \alpha'(d, a) - \gamma(d, a) \in \mathbb{R}^+.$$

Clearly, f is a continuous function in the variables d and a (see Equations 1–6). By Proposition 7, we can read the function g implicitly defined by the equation $f(d, g(d)) = 0$, but it is not easy to study the partial derived functions $\dfrac{\partial f}{\partial d}$ and $\dfrac{\partial f}{\partial a}$.

Lemma 8. *The function g is continuous.*

Proof. To show that g is a continuous function, we will fix a whichever element $\overline{x} \in U := (2.93, 2.97)$, and will prove that $\forall \varepsilon > 0$ there exists $\delta > 0$ such that if $|x - \overline{x}| < \delta$, $x \in U$, then $|g(x) - g(\overline{x})| < \varepsilon$.

For every $x \in U$, the function $f^x : y \to f(x, y)$ coincides with G_x, so that it is decreasing by Proposition 6; moreover $f(x, g(x)) = 0$.

Now, let $\overline{x} \in U$ and $\varepsilon > 0$ such that $(\overline{x}, g(\overline{x}) - \varepsilon)$ and $(\overline{x}, g(\overline{x}) + \varepsilon)$ belongs to the dominio of f- $(2.93, 2.97) \times (0, 2)$, then:

$$f(\overline{x}, g(\overline{x}) + \varepsilon) < 0 < f(\overline{x}, g(\overline{x}) - \varepsilon).$$

It follows by the continuity of f (respect to x) and by sign permanence theorem, that there exists $\delta > 0$ such that

$$f(x, g(\overline{x}) + \varepsilon) < 0 < f(x, g(\overline{x}) - \varepsilon), \quad \forall |x - \overline{x}| < \delta, x \in U.$$

As $f(x, g(x)) = 0$ and $f^x(x, y)$ is a strictly decreasing function, it follows that

$$g(\overline{x}) - \varepsilon < g(x) < g(\overline{x}) + \varepsilon, \quad \forall |x - \overline{x}| < \delta, x \in U.$$

The proof is complete. □

5. The main theorem

Before showing the final theorem we need two technical results that give the approximations of $g(d)$ and $d'(d, g(d)$ for some values d close to the extremes 2.93 and 2.97 of their domain. They are obtained by computations. For completeness we will give the whole proofs.

Lemma 9. *Put $d_1 = 2.9300001$. Then $a_1 = 1.3787 < g(d_1) < a_2 = 1.3794$. Moreover, $2.95528 < d'(d_1, g(d_1) < 2.95544$. In particular $d_1 < d'(d_1, g(d_1)$.*

Proof. The following relations can be obtain directly by computing $\gamma(d_1, a_1)$ and $\alpha'(d_1, a_1)$ using the Formulas 6, 8.

$$\begin{array}{rcl} 90.5506° < & \gamma(d_1, a_1) \simeq 90.55065° < & 90.5507° \\ 90.5630° < & \alpha'(d_1, a_1) \simeq 90.5631° < & 90.5632° \end{array}$$

and

$$\begin{array}{rcl} 90.5601° < & \gamma(d_1, a_2) \simeq 90.56011° < & 90.5602° \\ 90.5558° < & \alpha'(d_1, a_2) \simeq 90.55588° < & 90.5559° \end{array}$$

By Proposition 6, the continuous functions $\gamma(d_1, a)$ and $\alpha'(d_1, a)$ are respectively (strictly) increasing and decreasing, then we have the synchronization of $(\mathcal{Q}_{d_1,a}, \mathcal{Q}'_{d_1,a})$ when a lies in $(1.3787, 1.3794)$. Therefore

$$1.3787 < g(d_1) < 1.3794. \tag{18}$$

We will show now the second part of the Lemma. Looking at the equations 1 and 2 we see that $c(d)$ and $\delta(d)$ do not depend on a. Computing $c(d_1)$ and $\delta(d_1)$ we have the following bounds.

$$\begin{array}{rcl} 2.27023 < & c(d_1) \simeq 2.270234 < & 2.27024 \\ 69.1323^\circ < & \delta(d_1) \simeq 69.13239^\circ < & 69.1324^\circ \end{array}. \tag{19}$$

Clearly, the function $b(a) = \sqrt{a^2 + 3^2}$ is strictly increasing (see 3), thus by Equation 18 we have

$$3.301638 < \sqrt{9 + 1.3787^2} < b(g(d_1)) < \sqrt{9 + 1.3794^2} < 3.30192 \tag{20}$$

By Equation 4 the coordinates of $D' \in \mathcal{Q}'_{d_1, g(d_1)}$ are

$$u(d_1, g(d_1)) = -c(d_1)\sqrt{2}/2 + b(g(d_1)) \cos(\delta(d_1) - 45^\circ), \text{ and}$$
$$v(d_1, g(d_1)) = c(d_1)\sqrt{2}/2 + b(g(d_1)) \sin(\delta(d_1) - 45^\circ).$$

We can bound them using the inequalities 19 and 20:

$$\begin{aligned} & -2.27024\sqrt{2}/2 + 3.301638 \cos(24.1324^\circ) \\ & < u(d_1, g(d_1)) < -2.27023\sqrt{2}/2 + 3.30192 \cos(24.1323^\circ), \end{aligned} \tag{21}$$

thus,

$$1.4077 < u(d_1, g(d_1)) < 1.4081.$$

$$\begin{aligned} & 2.27023 \sqrt{2}/2 + 3.301638 \sin(24.1323^\circ) \\ & < v(d_1, g(d_1)) < 2.27024 \sqrt{2}/2 + 3.30192 \sin(24.1324^\circ), \end{aligned} \tag{22}$$

thus,

$$2.95515 < v(d_1, g(d_1)) < 2.9553.$$

Now we can bound $d'(d_1, g(d_1)) = \sqrt{(u(d_1, g(d_1)) - g(d_1))^2 + v(d_1, g(d_1))^2}$.

For let, we will consider the minimum (resp. the maximum) difference between $u(d_1, g(d_1))$ and $g(d_1)$, and the minimum (resp. the maximum) of the bound for $v(d_1, g(d_1))$ obtained above (precisely 18, 21 and 22). We have:

$$\sqrt{(1.4077 - 1.3794)^2 + 2.95515^2} < d'(d_1, g(d_1)) < \sqrt{(1.4081 - 1.3787)^2 + 2.9553^2}.$$

Then

$$2.95528 < d'(d_1, g(d_1)) < 2.95544.$$

□

Lemma 10. *Let* $d_2 = 2.9699999$. *Then* $a_1 = 1.3538 < g(d_2) < a_2 = 1.3541$. *Moreover,*

$$2.9124404 < d'(d_2, g(d_2) < 2.912513.$$

In particular $d_2 > d'(d_2, g(d_2)$.

Proof. We will proceed as in the above Lemma. So that as first step we will compute $\gamma(d_2, a_1)$ and $\alpha'(d_2, a_1)$ using the Formulas 6, 8:

$$\begin{array}{rcl} 91.0929^\circ < & \gamma(d_2, a_1) \simeq 91.093^\circ < & 91.0931^\circ \\ 91.0995^\circ < & \alpha'(d_2, a_1) \simeq 91.09956^\circ < & 91.0996^\circ \end{array}$$

and

$$\begin{array}{rcl} 90.5601^\circ < & \gamma(d_2, a_2) \simeq 90.56011^\circ < & 90.5602^\circ \\ 91.0958^\circ < & \alpha'(d_2, a_2) \simeq 91.09587^\circ < & 91.0959^\circ \end{array}$$

By Proposition 6, the continuous functions $\gamma(d_2, a)$ and $\alpha'(d_2, a)$ are respectively (strictly) increasing and decreasing, then we have the synchronization of $(\mathcal{Q}_{d_2,a}, \mathcal{Q}'_{d_2,a})$ when a lies in $(1.3538, 1.3541)$. Therefore

$$1.3538 < g(d_2) < 1.3541. \tag{23}$$

We will show now the second part of the Lemma. Looking at the equations 1 and 2 we see that $c(d)$ and $\delta(d)$ do not depend on a. Computing $c(d_2)$ and $\delta(d_2)$ we have the following bounds.

$$\begin{array}{rcl} 2.28478 < & c(d_2) \simeq 2.28479 < & 2.2848 \\ 68.19505^\circ < & \delta(d_2) \simeq 68.19506^\circ < & 68.19507^\circ \end{array}. \tag{24}$$

Now using the Equation 23, and taking into account that $b(d)$ is an increasing function, we have

$$\begin{aligned} 3.2913 &< \sqrt{9 + 1.3538^2} \simeq 3.291318 \\ &< b(g(d_2)) < \sqrt{9 + 1.3541^2} \simeq 3.291441 < 3.2915. \end{aligned} \tag{25}$$

By Equations 4 and 5 the coordinates of $D' \in \mathcal{Q}'_{d_2,g(d_2)}$ are

$u(d_2, g(d_2)) = -c(d_2)\sqrt{2}/2 + b(g(d_2)) \cos(\delta(d_2) - 45^\circ)$ and
$v(d_2, g(d_2)) = c(d_2)\sqrt{2}/2 + b(g(d_2)) \sin(\delta(d_2) - 45^\circ)$.

We can bound them using the inequalities (24) and (25):

$$\begin{aligned} &-2.2848\sqrt{2}/2 + \sqrt{9 + 1.3538^2} \cos(23.19507^\circ) \\ &< u(d, g(d)) < -2.28478\sqrt{2}/2 + \sqrt{9 + 1.3541^2} \cos(23.19505^\circ), \end{aligned} \tag{26}$$

thus

$$1.409681 < u(d_2, g(d_2)) < 1.40981;$$

$$\begin{aligned} &2.28478\,\sqrt{2}/2 + \sqrt{9 + 1.3538^2} \sin(23.19505^\circ) \\ &< v(d_2, g(d_2)) < 2.2848\,\sqrt{2}/2 + \sqrt{9 + 1.3541^2} \sin(23.19507^\circ) \end{aligned} \tag{27}$$

thus

$$2.91191 < v(d_2, g(d_2)) < 2.911974.$$

Now we can give a bound for
$d'(d_2, g(d)) = \sqrt{(u(d_2, g(d_2)) - g(d_2))^2 + v(d_2, g(d_2))^2}$.

For let, we will consider the minimum (resp. the maximum) difference between $u(d_2, g(d_2))$ and $g(d_2)$, and the minimum (resp. the maximum) of the bound for $v(d_2, g(d_2))$ obtained above (precisely 23, 26 and 27). We have:

$$\sqrt{(1.409681 - 1.3541)^2 + 2.91191^2}$$
$$< d'(d_2, g(d)) < \sqrt{(1.40981 - 1.3538)^2 + 2.911974^2}. \tag{28}$$

Then

$$2.9124404 < d'(d_2, g(d)) < 2.912513.$$

□

Theorem 11. *There exist* $\overline{d} \in (2.93, 2.97)$ *and* $\overline{a} \in (0, 2)$ *such that* $(\mathcal{Q}_{\overline{d},\overline{a}}, \mathcal{Q}'_{\overline{d},\overline{a}}) \in \mathcal{F}_{3,135^\circ}$ *is a pair of congruent-like quadrilaterals that are not congruent.*

Proof. Put $U = (2.93, 2.97)$ and $V = (0, 2)$. By Lemma 8 and by Relation 5 we have that:

$$g : d \in U \to g(d) \in V, \quad \text{and}$$
$$d' : (d, a) \in U \times V \to d'(d, a) \in \mathbb{R}$$

are continuous functions. It follows that

$$k : d \in U \to d'(d, g(d)) \in \mathbb{R}, \quad \text{and}$$
$$l : d \in U \to k(d) - d \in \mathbb{R}$$

are likewise continuous functions.

On the other hand by Lemmas 9 and 10, we have that there are two real numbers d_1 and d_2 such that:

$$d_1 < k(d_1) \quad \text{and} \quad d_2 > k(d_2).$$

It follows that l has a zero, and hence there exists $\overline{d}$ such that

$$k(\overline{d}) = \overline{d} = d'(\overline{d}, g(\overline{d})). \tag{29}$$

Now if we put $\overline{a} = g(\overline{d})$, then the quadrilaterals $\mathcal{Q}_{\overline{d},g(\overline{d})}$ e $\mathcal{Q}'_{\overline{d},g(\overline{d})}$ have all angles pairwise congruent, and also the sides AD e $A'D'$ are congruent (the other three sides are congruent by construction), thus they are congruent-like quadrilaterals.

Finally, if we put $\mathcal{Q}_{\overline{d},g(\overline{d})} = (A, B, C, D)$ and $\mathcal{Q}'_{\overline{d},g(\overline{d})} = (A', B', C', D')$, then $AB = a < 2 < CD < 2.93 < DA < 2.97 < 3$ (see Equation 13 and Theorem 11). On the other hand trivial geometric consideration yields: $3 < CB$. In particular the sides of maximum size in $\mathcal{Q}_{\overline{d},g(\overline{d})}$, are BC and DA, that are not consecutive. On the other hand by construction $BC = C'D'$ and by Equation 29 $DA = D'A'$, thus the sides of maximum size in $\mathcal{Q}'_{\overline{d},g(\overline{d})}$ are consecutive. Therefore $\mathcal{Q}_{\overline{d},g(\overline{d})}$ and $\mathcal{Q}'_{\overline{d},g(\overline{d})}$ are not congruent. The proof is complete. □

6. Conclusion

Congruence theorems for quadrilaterals (and more generally for polygons) could appear to be a difficult topic for many learners. The reasons for such difficulties relate to the complexities in learning to analyze the attributes of different quadrilaterals and to distinguish between critical and non-critical aspects (see [2] and [10]). We highlight that this kind of studies requires logical deduction, together with suitable interactions among concepts - of different areas of Mathematics - images and a certain skill in software computing.

We conclude by noting that the technique presented in section 2, is available also to define other pairs of congruent-like quadrilaterals that are not congruent, and hence the existence of other pairs (of congruent-like quadrilaterals that are not congruent) can be claimed. Finally we observe that Theorem 11 states the existence of pairs of non congruent quadrilaterals with all pieces pairwise congruent. Thus arise the following open questions:

1) Is it possible to characterize all pairs of quadrilaterals of such type?
2) Is it possible to find by elementary geometry (synthetic) a pair of non congruent quadrilaterals with all pieces pairwise congruent?

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