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A Remarkable Theorem on Eight Circles

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Abstract. For $i = 1, \dots, 6$, consider a closed chain of circles $\{\Gamma_i\}$ such that every two consecutive members Γ_i and Γ_{i+1} meet in the points (A_i, B_i), with indices modulo 6. We require that both sextuples $(A_1, \dots, A_6)$ and $(B_1, \dots, B_6)$ are cyclic. We prove the theorem that the three lines connecting the centers of the opposite circles of the chain concur. In the rest of the paper we present a slightly more general version of this theorem.

1. Introduction

We start from the following extension of Miquel's Six-Circles Theorem, which was given and proven in [9] (it is also mentioned in [10]).

Theorem 1 ([9]). *Let Γ_a and Γ_b be two circles. Let $n > 2$ be an even number, and take the points $A_1, \dots, A_n$ on Γ_a, and $B_1, \dots, B_n$ on Γ_b, such that each quadruple $(A_1, B_1, A_2, B_2), \dots, (A_{n-1}, B_{n-1}, A_n, B_n)$ is cyclic. Then the quadruple (A_n, B_n, A_1, B_1) is also cyclic.*

Note that this is an extension, indeed, since for $n = 4$ it returns Miquel's classical theorem. In the subsequent case, $n = 6$, the following remarkable concurrency occurs.

Theorem 2 (Szilassi). *For $i = 1, \dots, 6$, consider a closed chain of circles Γ_i in the plane such that two consecutive circles Γ_i and Γ_{i+1} intersect in the points (A_i, B_i) (with indices modulo 6). Denote the center of the ith circle of this chain by K_i. Suppose that the sextuple $(A_1, \dots, A_6)$ lies on a circle Γ_a, and likewise, the sextuple $(B_1, \dots, B_6)$ lies on a circle Γ_b different from Γ_a. Then the lines K_1K_4, K_2K_5 and K_3K_6 concur.*

This theorem was found by Lajos Szilassi when experimenting with the octuple of circles corresponding to Theorem 1 (cf. Figure 1). He communicated it to the author, but has given no proof [14]. In Section 2 we shall prove it. Section 3 is devoted to the background of this theorem, and in conclusion, we formulate a slightly generalized and extended version of it.

We note that there is a seemingly related theorem, due to Dao Thanh Oai, proved in several different ways [6, 2, 7]. It bears a formal resemblance to our theorem,

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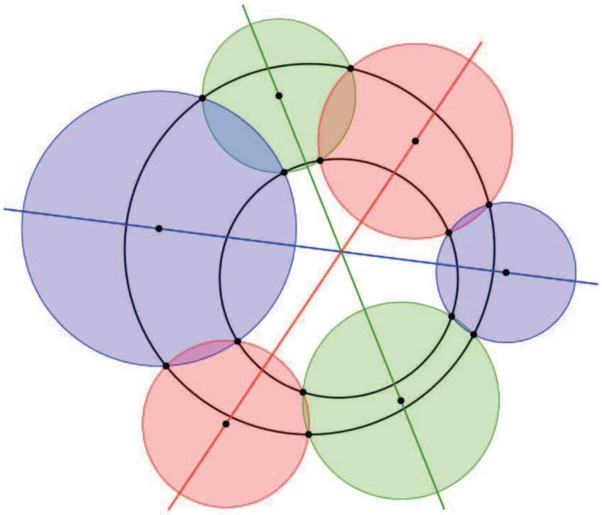

Figure 1. Illustration of Theorem 2 (drawn by Szilassi)

since it states the concurrence of lines connecting the opposite centers of a closed chain of six circles, like in our case (but with a different condition on these circles). Furthermore, a whole set of problems related to this theorem is posed in [8]. A careful scrutiny may reveal a closer connection between this theorem and ours; however, it is beyond the scope of this paper.

2. Proof of Theorem 2

In our proof, we shall use stereographic projection as well as duality in projective 3-space; so, to set the stage for this, we consider the following embeddings:

$$\mathbb{E}^2 \cup \{\infty\} \ \subset \ \mathbb{E}^2 \cup \{\ell_\infty\} \ \subset \ \mathbb{E}^3 \cup \{\pi_\infty\}. \tag{1}$$

In words, we use the Euclidean plane completed with the (unique) point at infinity, as the model of the inversive plane; it is embedded in the projective plane, whose model is the Euclidean plane completed with the line at infinity. This latter, in turn, is embedded in (actually, a restriction of) the projective 3-space, whose model is the Euclidean 3-space completed with the plane at infinity.

For the stereographic projection, we use the standard unit sphere $\mathbb{S}^2$, whose north pole N is the center of the projection, and the image plane is the tangent plane at the south pole S (for other details, see e.g. [4, 11, 13]). We also use an extension of this projection, from the same center and onto the same image plane, but with the whole projective 3-space as domain [13].

The dual transformation that we need is realized by polar reciprocity, whose reference sphere coincides with $\mathbb{S}^2$; in what follows, our term "duality" always refers to this reciprocity. Recall that in this case the dual of a point P is a plane P^* that passes through a point P' and is perpendicular to the radial line OP; here the point P' is the inverse of P, i.e. it lies on the ray from O to P at a distance $d(O, P') = d(O, P)^{-1}$.

The following statement is proved in [11], § 36.

Lemma 3 (Hilbert, Cohn-Vossen)**.** *Let Γ' be an arbitrary circle lying on the sphere and not passing through N. The planes tangent to the sphere at the points of Γ' envelop a circular cone, whose vertex we denote by V. Since Γ' does not pass through N, NV is not tangent to the sphere at N, and is therefore not parallel to the image plane; let M be the point at which NV intersects the image plane. The curve Γ that is the image of Γ' is a circle with M as center.*

Let (P, Q, R, S) be a cyclic quadruple of points lying on the sphere. Since it is cyclic, all the four points lie within a plane, which we denote by $\pi(P, Q, R, S)$. On the other hand, we denote by $V(P, Q, R, S)$ the vertex of the cone determined by the circumcircle of these points in the way described in the previous lemma. The following observation is a simple application of duality.

Proposition 4. *$\pi(P, Q, R, S)$ and $V(P, Q, R, S)$ are duals of each other.*

This implies, again by duality:

Proposition 5. *For two distinct cyclic quadruples of points (P_1, Q_1, R_1, S_1) and (P_2, Q_2, R_2, S_2) lying on the sphere, consider the line connecting $V(P_1, Q_1, R_1, S_1)$ and $V(P_2, Q_2, R_2, S_2)$. The dual of this line is the line of intersection of the planes $\pi(P_1, Q_1, R_1, S_1)$ and $\pi(P_2, Q_2, R_2, S_2)$.*

The following observation is implicit in the discussion of bundles of circles in [13] (but it can easily be seen as well).

Proposition 6. *Let Γ'_1 and Γ'_2 be two circles lying on the sphere, whose stereographic image are Γ_1 and Γ_2, respectively. The line of intersection of the planes of Γ'_1 and Γ'_2 projects from N precisely onto the radical line of Γ_1 and Γ_2.*

Note that this observation includes the case, too, when Γ'_1 and Γ'_2 are concentric; in this case the corresponding radical line coincides with the line at infinity.

The following lemma follows directly from Propositions 4 and 5, by a further application of duality.

Lemma 7. *Consider the following six quadruples of points lying on the sphere: $(P_1, Q_1, R_1, S_1), \ldots, (P_6, Q_6, R_6, S_6)$. For the point $V(P_i, Q_i, R_i, S_i)$ and plane $\pi(P_i, Q_i, R_i, S_i)$ $(i = 1, \ldots, 6)$ we use here the simplified notation V_i and π_i, respectively. Furthermore, we denote by ℓ_{14}, ℓ_{25}, ℓ_{36} the line connecting the pairs of points (V_1, V_4), (V_2, V_5), (V_3, V_6), respectively. Likewise, we denote by ℓ^*_{14}, ℓ^*_{25}, ℓ^*_{36} the line of intersection of the pairs of planes (π_1, π_4), (π_2, π_5) (π_3, π_6), respectively. Then, the lines $\ell_{14}, \ell_{25}, \ell_{36}$ are concurrent if and only if the lines $\ell^*_{14}, \ell^*_{25}, \ell^*_{36}$ lie within a common plane.*

Lemma 8. *Let Γ_1, Γ_2, Γ_4, Γ_5 be circles in the plane such that they are images under stereographic projection of the circles Γ'_1, Γ'_2, Γ'_4, Γ'_5, respectively, on the sphere. Suppose that the following conditions hold:*

(1) *Γ_1 and Γ_2 intersect in the points A_1, B_1;*
(2) *Γ_4 and Γ_5 intersect in the points A_4, B_4;*

(3) *the points* A_1, B_1, A_4, B_4 *form a cyclic quadruple.*

Denote by ℓ_{14} *the line of intersection of the planes* $\pi(\Gamma'_1)$ *and* $\pi(\Gamma'_4)$*, and by* ℓ_{25} *the line of intersection of the planes* $\pi(\Gamma'_2)$ *and* $\pi(\Gamma'_5)$ (*where* $\pi(\Gamma'_i)$ *denotes the plane of the circle* Γ'_i, $i = 1, 2, 4, 5$). *The lines* ℓ_{14} *and* ℓ_{25} *meet in a common point.*

Proof. Condition (1) implies that the radical line of Γ_1 and Γ_2 is A_1B_1. Hence we see by Proposition 6 that the planes $\pi(\Gamma'_1)$ and $\pi(\Gamma'_2)$ intersect in a line which meets the sphere of the stereographic projection in two points (actually, in the inverse image of A_1 and B_1). Denote this line by ℓ_{12}. Similarly, by Condition (2) we have a line, denoted by ℓ_{45}, which is the line of intersection of the planes $\pi(\Gamma'_4)$ and $\pi(\Gamma'_5)$.

Consider Condition (3). By the same reasoning as before, it implies that ℓ_{12} and ℓ_{45} lie within a common plane (this plane is precisely $\pi(A_1, B_1, A_4, B_4)$; cf. Figure 2). Hence ℓ_{12} and ℓ_{45} meet in a common point (recall that the plane in question is a projective plane, on account of (1)). This point is a common point of the planes $\pi(\Gamma'_1)$, $\pi(\Gamma'_2)$, $\pi(\Gamma'_4)$ and $\pi(\Gamma'_5)$. Thus it is the common point of the lines ℓ_{14} and ℓ_{25}, too. □

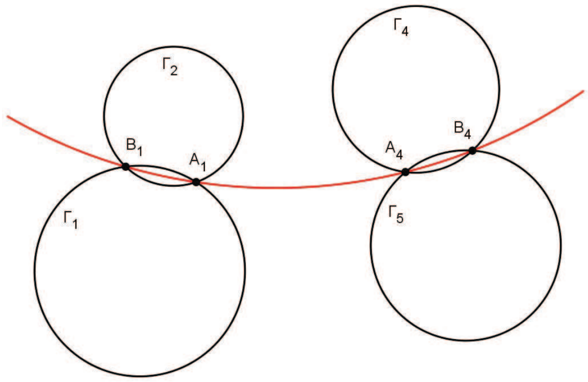

Figure 2. Arrangement of circles in Lemma 8

Lemma 9. *Consider the arrangement of points and circles given in Theorem 2. Each of the quadruples of points* (A_1, B_1, A_4, B_4), (A_2, B_2, A_5, B_5), (A_3, B_3, A_6, B_6) *is cyclic.*

Proof. Observe that the triple of circles $\Gamma_2, \Gamma_3, \Gamma_4$ closes to a four-membered chain through the circumcircle of the quadruple (A_1, B_1, A_4, B_4) such that these four circles, together with Γ_a and Γ_b, form a configuration corresponding to Miquel's classical six-circles theorem (cf. Figure 3). This means that Miquel's theorem implies the statement in this case. The other two cases can be verified analogously. □

Now we are ready to prove our theorem.

Proof of Theorem 2. Consider the following lines: $\ell_{14} = \pi(\Gamma'_1) \cap \pi(\Gamma'_4)$, $\ell_{25} = \pi(\Gamma'_2) \cap \pi(\Gamma'_5)$, $\ell_{36} = \pi(\Gamma'_3) \cap \pi(\Gamma'_6)$ (these lines exist, since we are in projective space). By Lemma 9, the quadruple (A_1, B_1, A_4, B_4) is cyclic. Thus the conditions

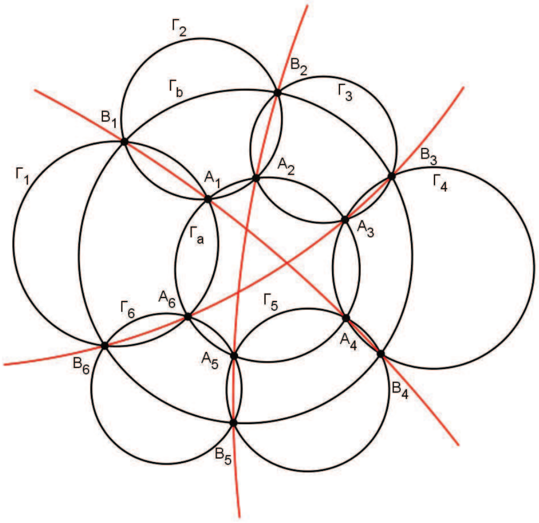

Figure 3. Completing the six-membered chain with three new circles

of Lemma 8 are fulfilled, hence the lines ℓ_{14} and ℓ_{25} meet in a common point (cf. Figures 2 and 3). By the same reasoning we obtain that each two of the lines ℓ_{14}, ℓ_{25} and ℓ_{36} have a common point. This means that they lie within the same plane. Denote their dual by ℓ^*_{14}, ℓ^*_{25}, and ℓ^*_{36}, respectively. By Lemma 7, these latter three lines connect the pairs of points (V_1, V_4), (V_2, V_5) and (V_3, V_6), respectively, and are concurrent. By Lemma 3, they project to the lines K_1K_4, K_2K_5 and K_3K_6, respectively. Thus it follows that those are also concurrent. □

3. Generalization and extension

The circumcircles of the quadruples (A_1, B_1, A_4, B_4), (A_2, B_2, A_5, B_5) and (A_3, B_3, A_6, B_6) play an important role in our proof. In this section we explore what is this role. To simplify the reference, we introduce the following notation: we denote by $\mathcal{C}(P, Q, R, S)$ the circumcircle of a quadruple of points (P, Q, R, S). Furthermore, we denote the circles in question as follows:

$$\Gamma''_3 = \mathcal{C}(A_1, B_1, A_4, B_4), \quad \Gamma''_1 = \mathcal{C}(A_2, B_2, A_5, B_5), \quad \Gamma''_2 = \mathcal{C}(A_3, B_3, A_6, B_6).$$

Remark 10. (a) Observe that the role of the Miquel type condition of Theorem 2 (i.e. the existence of the circumcircles Γ_a and Γ_b) is merely to ensure the existence of the circles Γ''_1, Γ''_2 and Γ''_3 (cf. the proof Lemma 9). Consequently, in our theorem the former condition can be replaced by the latter, weaker condition. Thus, with this weaker condition, we obtain a slightly more general version of the theorem.

(b) The weaker condition provides the possibility to extend the conclusion of the theorem, too. Namely, it turns out that not only one concurrence occurs. This will be formulated in Theorem 15 below.

To this end, first we change our former notation for the circles of the six-membered chain in Theorem 2 as follows: $\Gamma_1, \Gamma_3, \Gamma_5$ will be denoted by $\Gamma_1, \Gamma_2, \Gamma_3$, and $\Gamma_2, \Gamma_4, \Gamma_6$ will be denoted $\Gamma'_3, \Gamma'_1, \Gamma'_2$ respectively. Also, the centers will be denoted by K_i and K'_i, with indices according to the new notation of the circles.

With these new notations, observe that we have three triples of circles whose mutual relationship can be characterized as follows:

$$\begin{aligned} \Gamma''_1 &= \mathcal{C}((\Gamma_2 \cap \Gamma'_3) \cup (\Gamma_3 \cap \Gamma'_2)); \\ \Gamma''_2 &= \mathcal{C}((\Gamma_3 \cap \Gamma'_1) \cup (\Gamma_1 \cap \Gamma'_3)); \\ \Gamma''_3 &= \mathcal{C}((\Gamma_1 \cap \Gamma'_2) \cup (\Gamma_2 \cap \Gamma'_1)). \end{aligned} \tag{2}$$

Recall that the centers of circles sharing a common chord lie on the perpendicular bisector of this chord. Hence, the centers of the circles in (2) can be grouped in triples given by the following relations:

$$K''_1 = K_2K'_3 \cap K_3K'_2; \quad K''_2 = K_3K'_1 \cap K_1K'_3; \quad K''_3 = K_1K'_2 \cap K_2K'_1, \tag{3}$$

where K''_i denotes the center of the circle Γ''_i $(i = 1, 2, 3)$.

Observe the formal analogy of relations (2) and (3). Using the term *cross triangle* introduced by van Lamoen [12], relations (3) can be formulated as follows:

Proposition 11. *Triangle $K''_1K''_2K''_3$ is the cross triangle of triangles $K_1K_2K_3$ and $K'_1K'_2K'_3$.*

These relationships are illustrated in Figure 4. The following property can easily be checked.

Proposition 12. *Both triples of relations* (2) *and* (3) *are invariant under the following cyclic permutation: $X \mapsto X' \mapsto X'' \mapsto X$, where X stands for the circles in relations* (2) *as well as for the corresponding centers in* (3).

As a consequence, Proposition 11 can be formulated in the following seemingly stronger, but in fact equivalent version.

Proposition 11$'$. *Any of the triangles $K_1K_2K_3$, $K'_1K'_2K'_3$, $K''_1K''_2K''_3$ is the cross triangle of the other two triangles.*

We apply the following theorem of van Lamoen [12]:

Theorem 13 (van Lamoen)**.** *Let ABC, $A'B'C'$ and $A''B''C''$ be three triangles such that $A''B''C''$ is the cross triangle of ABC and $A'B'C'$. Suppose that ABC and $A'B'C'$ are centrally perspective. Then ABC and $A''B''C''$ are centrally perspective, and likewise, $A'B'C'$ and $A''B''C''$ are also centrally perspective. Moreover, the three perspectors are collinear.*

Note that here any two perspectivities imply the third, on account of Proposition 12. Observe that the three triangles in this theorem determine a configuration of type $(12_4, 16_3)$ which consists of

- the nine vertices of the triangles;
- the three perspectors;
- the six lines determining the cross triangle relationship;
- the nine projecting lines meeting by threes at the three perspectors;

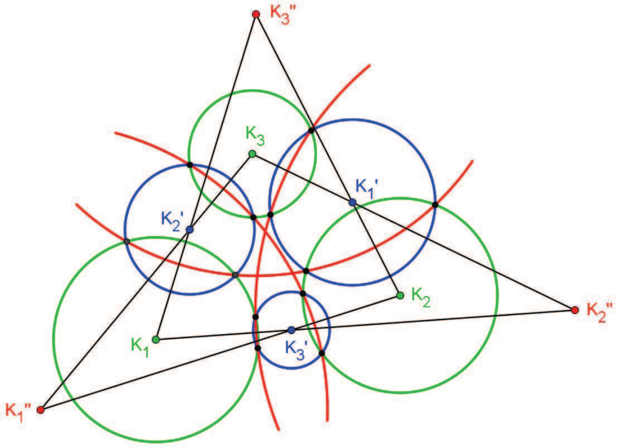

Figure 4. The desmic system of nine circles

- the line connecting these perspectors.

It is called the *desmic configuration*, and the triangles are called the *desmic mates* of each other [5]. This is a famous configuration which goes back to the work of Hesse and Salmon in the 19th century [3]. Moreover, as one can check in Figure 5, it is isomorphic with the Reye configuration (cf. [11], § 22). This latter, in turn, relates it with the celebrated *desmic tetrahedra* of Stephanos [1] (hence the term which we are using here).

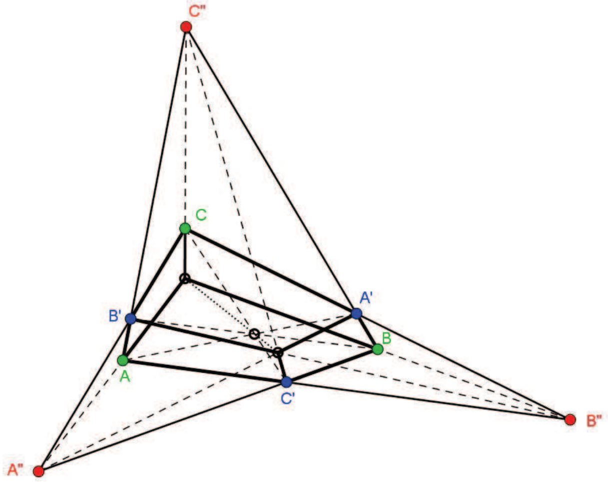

Figure 5. The desmic configuration of type $(12_4, 16_3)$. (A combinatorial cube is indicated showing the isomorphism with the Reye configuration)

Definition 14. *For $i = 1, 2, 3$, let Γ_i, Γ'_i, Γ''_i be nine circles which satisfy the relations (2) above such that each of the intersections $\Gamma_i \cap \Gamma'_j$ $(i \neq j)$ consists* *of precisely two points. Suppose, furthermore, that the centers* K_i*,* K_i'*,* K_i'' *of these circles are the vertices of triangles* $K_1K_2K_3$*,* $K_1'K_2'K_3'$*,* $K_1''K_2''K_3''$ *which are desmic mates of each other. Then we call these circles a* desmic system of nine circles*.*

Using this definition, and taking into consideration Remark 10a as well as Theorem 13, we obtain the following slightly more general version of Theorem 2.

Theorem 15. *For* $i = 1, \ldots, 6$*, consider a closed chain of circles* $\{\Gamma_i\}$ *such that every two consecutive members* Γ_i *and* Γ_{i+1} *meet in the points* (A_i, B_i)*, with indices modulo 6. Suppose that this chain can be completed with three circles which are circumscribed on the quadruples* (A_1, B_1, B_4, A_4)*,* (A_2, B_2, B_5, A_5) *and* (A_1, B_1, B_6, A_6)*. Then these nine circles form a desmic system.*

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