Forum Geometricorum
Volume 18 (2018) 409–412.

FORUM GEOM
ISSN 1534-1178

An Analogue to Pappus Chain theorem with Division by Zero

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Abstract. Considering a line passing through the centers of two circles in a Pappus chain, we give a theorem analogue to Pappus chain theorem.

1. Introduction

Let α, β and γ be circles with diameters BC, CA and AB, respectively for a point C on the segment AB (see Figure 1). Pappus chain theorem says: if $\{\alpha = \delta_0, \delta_1, \delta_2, \cdots\}$ is a chain of circles whose members touch the circles β and γ, the distance between the center of the circle δ_n and the line AB equals $2nr_n$, where r_n is the radius of δ_n. In this article we show that if we consider a line passing through the centers of two circles in the chain instead of AB, a similar theorem still holds including the case in which the two circles are symmetric in the line AB.

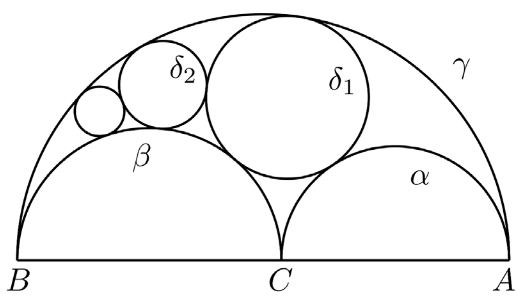

Figure 1.

2. Results

Let a and b be the radii of the circles α and β, respectively. We use a rectangular coordinate system with origin C such that A and B have coordinates $(-2b, 0)$ and $(2a, 0)$, respectively. Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{\alpha, \beta, \gamma\}$. We consider the chain of circles $\mathcal{C} = \{\cdots, \delta_{-2}, \delta_{-1}, \varepsilon_1 = \delta_0, \delta_1, \delta_2, \cdots\}$ whose members touch the circles ε_2 and ε_3, where we assume that δ_i lies on the region $y > 0$ if $i > 0$ (see Figure 2).

If $\varepsilon_1 = \alpha$, the chain is explicitly denoted by $\mathcal{C}_\alpha$. The chains $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$ are defined similarly. Let $c = a + b$ and let (x_n, y_n) and r_n be the coordinates of the center and the radius of the circle δ_n. Pappus chain theorem also holds for the chains $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$, i.e., we have $y_n = 2nr_n$, and x_n and r_n are given in Table 1 [3, 4].

Publication Date: December 6, 2018. Communicating Editor: Paul Yiu.

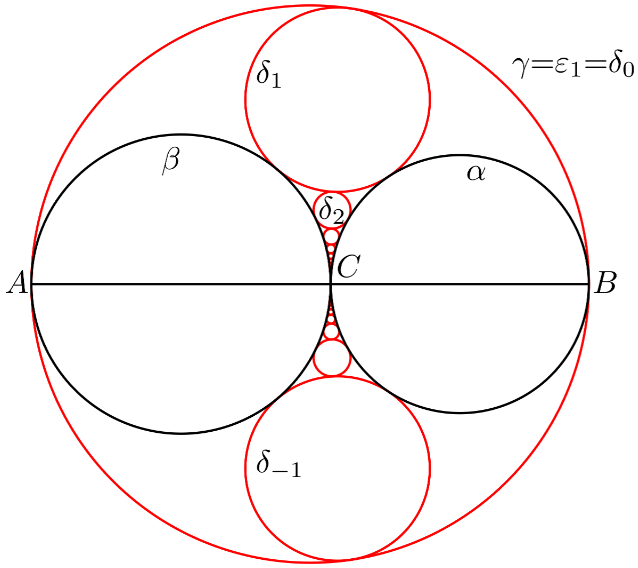

Figure 2: $\mathcal{C}_\gamma, \varepsilon_1 = \gamma,\ \{\varepsilon_2, \varepsilon_3\} = \{\alpha,\ \beta\}$

Chain	x_n	r_n
$\mathcal{C}_\alpha$	$-2b + \dfrac{bc(b+c)}{n^2a^2+bc}$	$\dfrac{abc}{n^2a^2+bc}$
$\mathcal{C}_\beta$	$2a - \dfrac{ca(c+a)}{n^2b^2+ca}$	$\dfrac{abc}{n^2b^2+ca}$
$\mathcal{C}_\gamma$	$\dfrac{ab(b-a)}{n^2c^2-ab}$	$\dfrac{abc}{n^2c^2-ab}$

Table 1: $y_n = 2nr_n$

Let $l_{i,j}$ $(i \neq j)$ be the line passing through the centers of the circles δ_i and δ_j for $\delta_i, \delta_j \in \mathcal{C}$. It is expressed by the equations

$$\left\{\begin{array}{l} 2(bc - a^2ij)x + a(b+c)(i+j)y - 2b(2a^2ij - c(b-c)) = 0, \\ 2(ca - b^2ij)x - b(c+a)(i+j)y + 2a(2b^2ij + c(c-a)) = 0, \\ 2(ab + c^2ij)x + c(a-b)(i+j)y - 2ab(a-b) = 0 \end{array}\right. \tag{1}$$

for $\mathcal{C}_\alpha$, $\mathcal{C}_\beta$, $\mathcal{C}_\gamma$, respectively.

Let $H_{i,j}(n)$ be the point of intersection of the lines $x = x_n$ and $l_{i,j}$ with y-coordinate $h_{i,j}(n)$. Let $d_{i,j}(n) = h_{i,j}(n) - y_n$, i.e., $d_{i,j}(n)$ is the signed distance between the center of δ_n and $H_{i,j}(n)$. The following theorem is an analogue to Pappus chain theorem. It is also a generalization of [1] (see Figure 3).

Theorem 1. *If $i + j \neq 0$, then $d_{i,j}(n) = f_{i,j}(n)r_n$ holds, where*

$$f_{i,j}(n) = \frac{2(n-i)(n-j)}{i+j} = 2\left(\frac{n^2}{i+j} - n + \frac{ij}{i+j}\right). \tag{2}$$

Proof. We consider the chain $\mathcal{C}_\alpha$. By Table 1 and (1), we get

$$h_{i,j}(n) = \frac{2(n^2+ij)abc}{(i+j)(n^2a^2+bc)} = 2\frac{(n^2+ij)}{(i+j)}r_n.$$

Therefore

$$d_{i,j}(n) = h_{i,j}(n) - y_n = 2\frac{(n^2+ij)}{(i+j)}r_n - 2nr_n = \frac{2(n-i)(n-j)}{i+j}r_n.$$

The rest of the theorem can be proved in a similar way. □

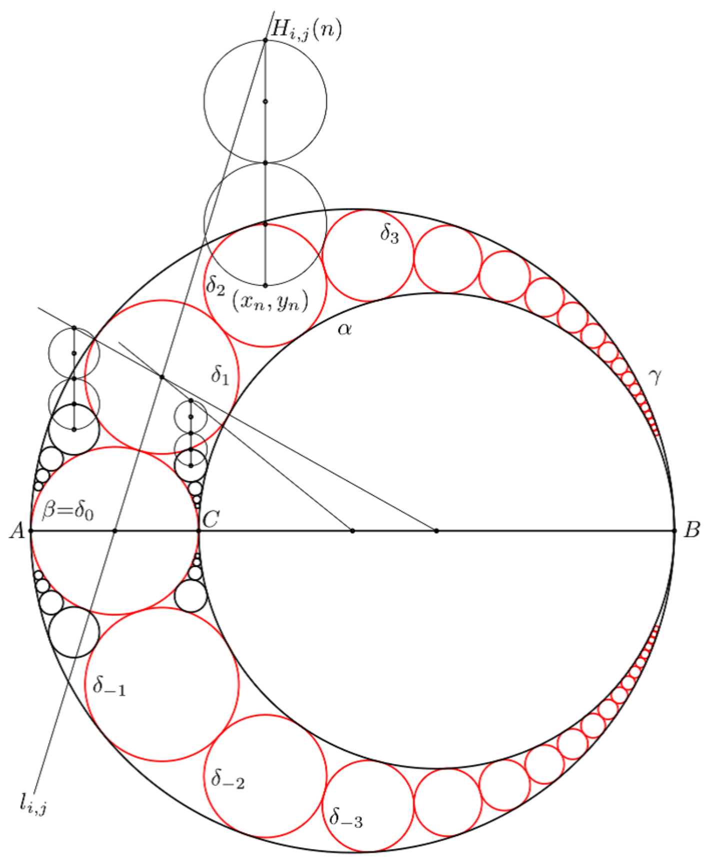

Figure 3: $\mathcal{C}_\beta,\ \{i,\ j\} = \{0,\ 1\},\ n = 2$

Corollary 2. *If $i = 0$ in Theorem 1, the following statements hold.*
(i) *If $j = \pm 1$, $d_{i,j}(n) = \pm 2n(n \mp 1)r_n$.*
(ii) *If $j = \pm 2$, $d_{i,j}(n) = \pm n(n \mp 2)r_n$.*

Corollary 3. $d_{i,j}(n) - d_{i,j}(-n) = -4nr_n$ *for any integers i, j, n with $i \neq \pm j$.*

3. The case $i + j = 0$

We consider Theorem 1 in the case $i + j = 0$ under the definition of division by zero: $z/0 = 0$ for any real number z [2]. In this case the line $l_{i,j}$ is perpendicular to AB. Since $i + j = 0$ implies $f_{i,j}(n) = -2n$ by (2), we get the same conclusion as to Pappus chain theorem for $l_{i,j}$.

Theorem 4. *If* $i + j = 0$, $d(n) = -2nr_n$ *holds.*

Since two parallel lines meet in the origin [5], $H_{i,j}(n)$ coincides with the origin, if $i + j = 0$, i.e., $h_{i,j}(n) = 0$. Hence Theorem 4 can also be derived from this fact with Pappus chain theorem. The theorem shows that *the distances from the center of the circle* δ_n *to the two lines* AB *and* $x = x_i$ *are the same for any integer* i. This is one of the unexpected phenomena for perpendicular lines derived from the definition of the division by zero. For another such example see [6]. Since Theorem 1 can be stated even in the case $i + j = 0$, we get :

Theorem 5. $d_{i,j}(n) = f_{i,j}(n)r_n$ *holds for any integers* i, j, n, *where* $i \neq j$.

Now Corollary 3 also holds even in the case $i + j = 0$.

Corollary 6. $d_{i,j}(n) - d_{i,j}(-n) = -4nr_n$ *for any integers* i, j, n *with* $i \neq j$.

4. Conclusion

The recent definition, $z/0 = 0$ for a real number z, yields several unexpected phenomena, which are especially significant for perpendicular lines. In this paper we get one more such result in section 3, for which we are still looking for a suitable interpretation.

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