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Polygons on the Sides of an Octagon

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Abstract. Van Aubel's theorem has some interesting generalizations. Some were dealt with by Krishna (2018a and 2018b). In this article, we intend, encouraged by the works cited, to prove a new generalization of Van Aubel's theorem, which consists of the construction of a parallelogram from an octagon surrounded by regular n-sided polygons.

The present article will demonstrate the following result about octagons, driven by the results of Krishna [1, 2].

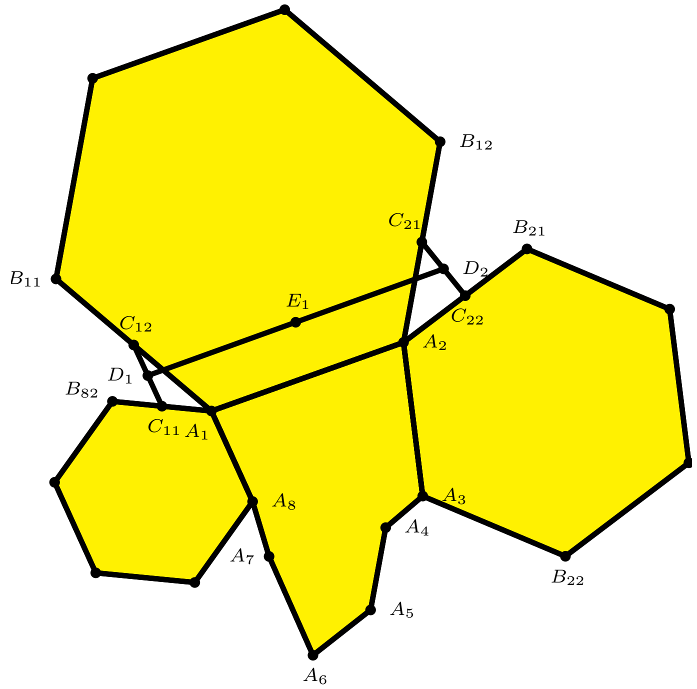

Figure 1

Theorem 1. *Take a random octagon O : $A_1A_2\cdots A_8$, either convex or concave. Having fixed an integer $n \geq 3$, consider the eight regular polygons of n sides constructed externally on the sides of O. Denote the sixteen sides of the eight*

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polygons that have the common vertex A_j with the octagon, $j = 1, \ldots, 8$, by A_1B_{82}, A_1B_{11}, A_2B_{12}, A_2B_{21}, $\ldots$, A_8B_{72}, and A_8B_{81}, as illustrated in Figure 1. Denote the midpoints of these sides by C_{11}, C_{12}, C_{21}, C_{22}, $\ldots$, C_{81}, and C_{82}, respectively. The figure shows only three of the regular polygons so that the image is not overloaded.

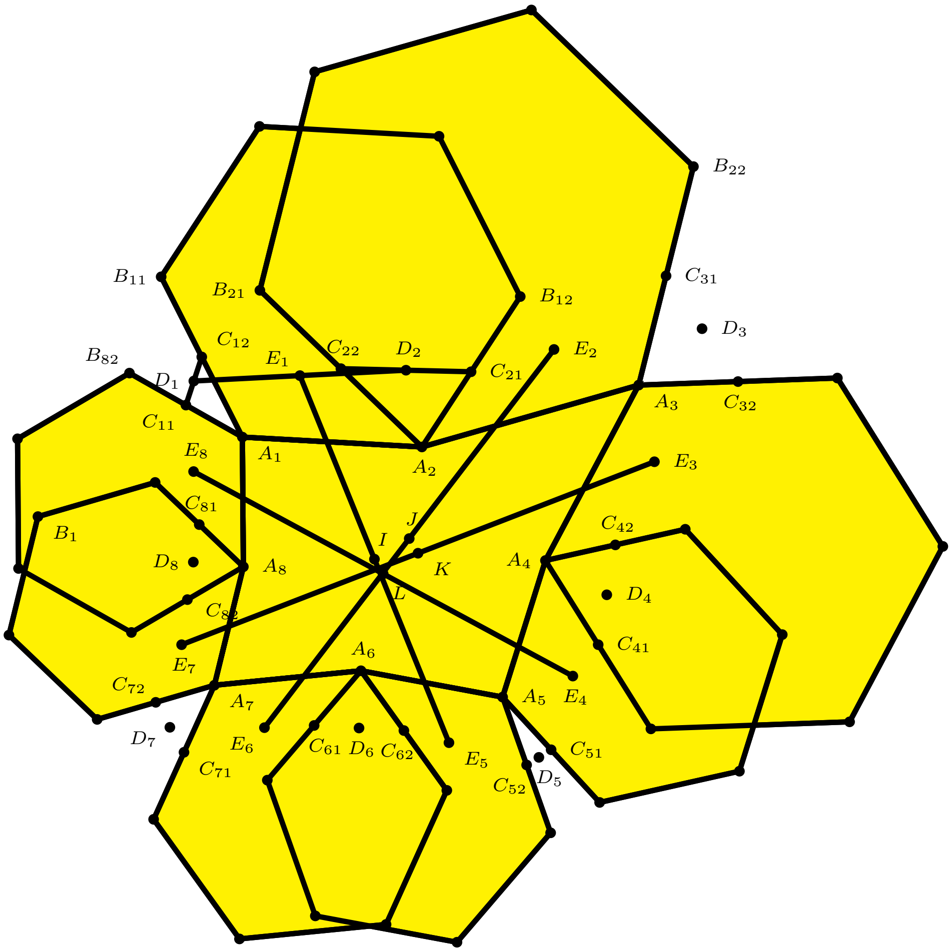

Figure 2

Also consider the midpoints of the segments $C_{j1}C_{j2}$:

$$D_j = \frac{C_{j1} + C_{j2}}{2}, \qquad j = 1, \ldots, 8 \pmod 8$$

and the midpoints of D_jD_{j+1}:

$$E_j = \frac{D_j + D_{j+1}}{2}.$$

.

Under these conditions, the midpoints of E_jE_{j+4}, $j = 1, \ldots, 8 \pmod 8$, *denoted by* I, J, K, *and* L, *respectively, form a parallelogram, as shown in Figure 2.*

Proof. Let us demonstrate by denoting the points I, J, K, and L as functions of the complex numbers corresponding to the vertices of the octagon and the internal angle of the polygon built on the side. Consider $A_j = (x_j,\ y_j)$, $j = 1, 2, \ldots, 8$ for each vertex of the octagon in the Cartesian Plane. Denote the complex number corresponding to each A_j by $a_j = x_j + iy_j$, where $i^2 = -1$. Similarly, denote the complex numbers corresponding to the points B_j, C_j, D_j, and E_j by b_j, c_j, d_j and e_j, respectively. Take M_j as the midpoint of the segment A_jA_{j+1}, so that $m_j = \frac{a_j + a_{j+1}}{2}$ is the complex number corresponding to M_j.

Considering $\hat{A} = \widehat{B_{11}A_1A_2}$ the internal angle of each of the eight regular polygons. Then, through the hypotheses, we can conclude that $c_{12} - a_1$ is equal to the complex number resulting from the rotation of $m_1 - a_1$ by $\hat{A}$ degrees counterclockwise. Namely,

$$c_{12} - a_1 = (m_1 - a_1)(\cos \hat{A} + i \sin \hat{A}).$$

From now on let us denote $\operatorname{cis}(\hat{A}) = \cos \hat{A} + i \sin \hat{A}$. Then,

$$c_{12} = a_1 + (m_1 - a_1)\operatorname{cis}(\hat{A}).$$

The complex number $c_{11} - a_1$ is equal to the complex number resulting from the rotation of $m_8 - a_1$ by $\hat{A}$ degrees clockwise, or $360° - \hat{A}$ counterclockwise, that is,

$$\begin{aligned}
& c_{11} - a_1 = (m_8 - a_1)\operatorname{cis}(360° - \hat{A}) \\
\Longrightarrow\ & c_{11} - a_1 = (m_8 - a_1)(\cos(360° - \hat{A}) + i \sin(360° - \hat{A})) \\
&\qquad = (m_8 - a_1)(\cos \hat{A} - i \sin \hat{A}) \\
&\qquad = (m_8 - a_1)(\cos(-\hat{A}) + i \sin(-\hat{A})) \\
&\qquad = (m_8 - a_1)\operatorname{cis}(-\hat{A}) \\
\Longrightarrow\ & c_{11} = a_1 + (m_8 - a_1)\operatorname{cis}(-\hat{A}).
\end{aligned}$$

Therefore, we can find

$$\begin{aligned}
d_1 =\ & \frac{c_{11} + c_{12}}{2} \\
=\ & \frac{1}{2}\left(a_1 + (m_8 - a_1)\operatorname{cis}(-\hat{A}) + a_1 + (m_1 - a_1)\operatorname{cis}(\hat{A})\right) \\
=\ & 1/2\left(a_1 + \left(\frac{a_8 + a_1}{2} - a_1\right)\operatorname{cis}(-\hat{A}) + a_1 + \left(\frac{a_1 + a_2}{2} - a_1\right)\operatorname{cis}(\hat{A})\right) \\
=\ & 1/2\left(2a_1 + \frac{a_8 - a_1}{2}\operatorname{cis}(-\hat{A}) + \frac{a_2 - a_1}{2}\operatorname{cis}(\hat{A})\right).
\end{aligned}$$

Similarly, we can find

$$\begin{aligned} d_2 = &\ \frac{c_{21} + c_{22}}{2} \\ = &\ \frac{1}{2}\left(2a_2 + \frac{a_1 - a_2}{2}\operatorname{cis}(-\hat{A}) + \frac{a_3 - a_2}{2}\operatorname{cis}(\hat{A})\right). \end{aligned}$$

Then

$$\begin{aligned} e_1 = &\ \frac{d_1 + d_2}{2} \\ = &\ \frac{1}{4}\left(2a_1 + \frac{a_8 - a_1}{2}\operatorname{cis}(-\hat{A}) + \frac{a_2 - a_1}{2}\operatorname{cis}(\hat{A})\right. \\ &\qquad \left. +2a_2 + \frac{a_1 - a_2}{2}\operatorname{cis}(-\hat{A}) + \frac{a_3 - a_2}{2}\operatorname{cis}(\hat{A})\right) \\ = &\ \frac{1}{4}\left(2(a_1 + a_2) + \frac{a_8 - a_2}{2}\operatorname{cis}(-\hat{A}) + \frac{a_3 - a_1}{2}\operatorname{cis}(\hat{A})\right). \end{aligned}$$

Similarly, we obtain

$$e_5 = \frac{1}{4}\left(2(a_5 + a_6) + \frac{a_4 - a_6}{2}\operatorname{cis}(-\hat{A}) + \frac{a_7 - a_5}{2}\operatorname{cis}(\hat{A})\right).$$

Thus,

$$\begin{aligned} I = &\ \frac{e_1 + e_5}{2} \\ = &\ \frac{1}{8}\left(2(a_1 + a_2 + a_5 + a_6) + \frac{a_8 + a_4 - a_2 - a_6}{2}\operatorname{cis}(-\hat{A})\right. \\ &\qquad \left. + \frac{a_3 + a_7 - a_1 - a_5}{2}\operatorname{cis}(\hat{A})\right). \end{aligned}$$

Similarly, we can obtain:

$$\begin{aligned} J = &\ \frac{e_2 + e_6}{2} \\ = &\ \frac{1}{8}\left(2(a_2 + a_3 + a_6 + a_7) + \frac{a_1 + a_5 - a_3 - a_7}{2}\operatorname{cis}(-\hat{A})\right. \\ &\qquad \left. + \frac{a_4 + a_8 - a_2 - a_6}{2}\operatorname{cis}(\hat{A})\right), \\ K = &\ \frac{e_3 + e_7}{2} \\ &\ \frac{1}{8}\left(2(a_3 + a_4 + a_7 + a_8) + \frac{a_2 + a_6 - a_4 - a_8}{2}\operatorname{cis}(-\hat{A})\right. \\ &\qquad \left. + \frac{a_5 + a_1 - a_3 - a_7}{2}\operatorname{cis}(\hat{A})\right), \end{aligned}$$

and

$$\begin{aligned} L = & \ \frac{e_4 + e_8}{2} \\ = & \ \frac{1}{8}\left(2(a_4 + a_5 + a_8 + a_1) + \frac{a_3 + a_7 - a_5 - a_1}{2}\mathrm{cis}(-\hat{A})\right. \\ & \left. + \frac{a_6 + a_2 - a_4 - a_8}{2}\mathrm{cis}(\hat{A})\right). \end{aligned}$$

Anyway, let us calculate

$$\begin{aligned} |J - K| = & \ \frac{1}{8}\Big| \, 2(a_2 + a_6 - a_4 - a_8) \\ & + \frac{a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8}{2}\mathrm{cis}(-\hat{A}) \\ & + \frac{-a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8}{2}\mathrm{cis}(\hat{A})\Big|. \end{aligned}$$

Remembering that

$$\begin{aligned} I = & \ \frac{1}{8}\left(2(a_1 + a_2 + a_5 + a_6) + \frac{a_8 + a_4 - a_2 - a_6}{2}\mathrm{cis}(-\hat{A})\right. \\ & \left. + \frac{a_3 + a_7 - a_1 - a_5}{2}\mathrm{cis}(\hat{A})\right) \end{aligned}$$

and

$$\begin{aligned} L = & \ \frac{1}{8}\left(2(a_4 + a_5 + a_8 + a_1) + \frac{a_3 + a_7 - a_5 - a_1}{2}\mathrm{cis}(-\hat{A})\right. \\ & \left. + \frac{a_6 + a_2 - a_4 - a_8}{2}\mathrm{cis}(\hat{A})\right), \end{aligned}$$

we have:

$$\begin{aligned} |I - L| = & \ \frac{1}{8}\Big| \, 2(a_2 + a_6 - a_4 - a_8) \\ & + \frac{a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8}{2}\mathrm{cis}(-\hat{A}) \\ & + \frac{-a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8}{2}\mathrm{cis}(\hat{A})\Big| \\ = & \ |J - K|. \end{aligned}$$

In an analogous way, it is possible to prove that

$$|K - L| = |J - I|.$$

This concludes the demonstration. □

References

[1] D. N. V. Krishna, An equilateral triangle associated with a hexagon, *Math. Gazette*, 102 (555) (2018a) 515–516.

[2] D. N. V. Krishna, A note on special cases of Van Aubel's theorem, *International Jounal of Advances in Applied Mathematics and Mechanics*, 5(4) (2018b) 30–51.

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