December 8th, 2003

2003 Putnam A5.

Solution. Let D_n be the set of Dyck n-paths, A_n be the subset of D_n with no return of even length, $B_n = D_n - A_n$, and C_n be the subset of B_n with an even return at the very end (i.e., a return of even length to $(2n, 0)$). We use lower-case letters to mean the cardinality of its corresponding set in upper-case letters.

Firstly, $B_n = \cup_{k=1}^{n} B_n(k)$, where $B_n(k)$ is the subset of B_n with last return of even length at $(2k, 0)$. Then $b_n(k) = c_k a_{n-k}$ for each k. Thus $b_n = \sum_{k=1}^{n} c_k a_{n-k}$, where $a_0 = 1$.

Next, $A_{n+1} = \cup_{k=0}^{n} A_{n+1}(k)$, where $A_{n+1}(k)$ is the subset of A_{n+1} that dip back (with returns of odd lengths, of course) to the x-axis for the first time at $(2k+2, 0)$. Note that before returning to the x-axis, each path in $A_{n+1}(k)$ must stayed or above the level of $y = 1$ and hit $(2k+1, 1)$ with a return of even length. Thus $a_{n+1}(k) = c_k a_{n-k}$. Hence $a_{n+1} = \sum_{k=0}^{n} c_k a_{n-k}$, where $c_0 = 1$.

Comparing the sums of b_n and a_{n+1}, we see that $a_{n+1} = a_n + b_n = d_n$.

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