POLYA008: Discussion/Solution. I prove, rigorously I hope, that *if* m *is large enough*, then $N_{m+1} \in \{N_m + 2, N_m + 3\}$, where the integer N_m is defined to be the first integer n for which $m^n < n!$.

It is also easy to decide what "large enough" means; i.e., how large m should be. I believe the estimates hold already for $m \geq 18$; since the cases in which $m \leq 17$ can be verified by direct computation, the result would hold for all positive integers m.

We use the following version of Stirling's formula (cf [1][12.5]):

$$\Gamma(t) = e^{-t} t^{t-1/2} \sqrt{2\pi} e^{\theta(t)/t}$$

valid for all $t > 0$ where $\theta : (0, \infty) \to [0, \frac{1}{12}]$.

Let m be a positive integer and let $n = N_m$ so that $m^n < n!$ while $m^{n-1} \geq (n-1)!$ (the only case in which we can have equality is $m = 1$). We begin with $m^n < n! = n\Gamma(n)$, use Stirling's formula for $t = n$ to evaluate the right hand side, and take the n-th root of both sides to get

$$m < e^{-1} n \left(2\pi n\right)^{1/2n} e^{\theta(n)/n^2} \tag{1}$$

Writing $(n+3)! = (n+3)\Gamma(n+3)$, using Stirling's formula for $t = n+3$, and taking the $n+3$-rd root, we see that

$$\begin{aligned}(n+3)!^{1/(n+3)} &= e^{-1}(n+3)(2\pi(n+3))^{1/(2(n+3)} e^{\theta(n+3)/(n+3)^2} \\ &\geq e^{-1}(n+3)(2\pi(n+3))^{1/(2(n+3)}.\end{aligned}$$

Let $x = \log(2\pi(n+3))/(2(n+3))$ so that

$$(2\pi(n+3))^{1/(2(n+3)} = e^x > 1 + x = 1 + \frac{\log(2\pi(n+3))}{2(n+3)},$$

hence

$$(n+3)!^{1/(n+3)} > \frac{n+3}{e} + \frac{\log[2\pi(n+3)]}{2e}. \tag{2}$$

From (1) and the fact that $\theta(t) \leq 12$ for all $t > 0$,

$$m < e^{-1} n \left(2\pi n\right)^{1/2n} e^{12/n^2} = \frac{n}{e} e^z$$

where $z = (1/2n)\log(2\pi n) + (12/n^2)$. By Taylor's formula,

$$\begin{aligned} m &< \frac{n}{e}\left(1 + z + \frac{1}{2}z^2 e^z\right) = \frac{n}{e} + \frac{\log(2\pi n)}{2e} + \frac{12}{ne} + \frac{1}{2} n z^2 e^{z-1} \\ &= \frac{n}{e} + \frac{\log(2\pi n)}{2e} + \frac{12}{ne} + O(\frac{\log^2 n}{n}) = \frac{n}{e} + \frac{\log(2\pi n)}{2e} + O(\frac{\log^2 n}{n}) \end{aligned}$$

Subtracting from (2) we get

$$(n+3)!^{1/(n+3)} - m > \frac{3}{e} + \frac{\log(\frac{n+3}{n})}{2e} - O(\frac{logn}{n}) > \frac{3}{e} - O(\frac{logn}{n}).$$

Since $3/e > 1$, and since clearly $n = N_m > m$, it follows that for m large enough we have

$$(n+3)!^{1/(n+3)} - m > 1; \;\text{ i.e., }\; (n+3)! > (m+1)^{n+3}.$$

This proves that $N_{m+1} \leq N_m + 3$ if m is large enough. In trying to determine how large is "large enough," we can be more careful in estimating the term that appears as $O(\log^2 n/n)$. I get that for $n = 45$ (which roughly corresponds to $m = 18$) the term is < 0.1. Because $(3/e) - 1 > 0.1$, this would prove that $N_{m+1} \leq N_m + 3$ for $m \geq 18$, and it is easily verified to hold by direct computations for $m \leq 17$.

The proof that $N_{m+1} \geq N_m + 2$ is similar. From

$$m^{n-1} > (n-1)! = (n-1)\Gamma(n-1)$$

we get applying Stirling's formula with $t = n-1$ and taking the $(n-1)$-st root,

$$\begin{aligned} m &> e^{-1}(n-1)(2\pi(n-1))^{1/2(n-1)}e^{\theta(n-1)/(n-1)^2} \\ &\geq e^{-1}(n-1)(2\pi(n-1))^{1/2(n-1)}. \end{aligned} \tag{3}$$

By Stirling, with $t = n+1$ applied to $(n+1)! = (n+1)\Gamma(n+1)$, after taking the (n-1)-st root, we get

$$\begin{aligned} (n+1)!^{1/(n+1)} &= e^{-1}(n+1)(2\pi(n+1))^{1/2(n+1)}e^{\theta(n+1)/(n+1)^2} \\ &\leq e^{-1}(n+1)(2\pi(n+1))^{1/2(n+1)}e^{12/(n+1)^2}. \end{aligned}$$

Subtracting from (3), and proceeding very much as we did before to prove $N_{M+1} \leq N_m + 3$, we get

$$m - (n+1)!^{1/(n+1)} > \frac{n-1}{e} - \frac{n+1}{e} - O(\frac{\log^2 n}{n}) = -\frac{2}{e} - O(\frac{\log^2 n}{n}).$$

Since $2/e < 1$, we proved that

$$m + 1 > (n+1)!^{1/(n+1)}; \;\text{ i.e. }\; (m+1)^{n+1} > (n+1)!,$$

hence $N_{m+1} > n + 1 = N_m + 1$, if m is large enough. Once more, it is easy to estimate how large m has to be so that it is "large enough," and one can deal with the smaller cases by direct computation.

References

[1] E.T. Whittaker and G.N. Watson, *Modern Analysis*, Cambridge University Press, Cambridge 1952.