Putnam 2002

A1. It is easy to prove by induction that

$$P_n(x) = (-1)^n n! k^n x^{n(k-1)} + Q_n(x)$$

where $Q_n(1) = 0$. It follows that

$$\boxed{P_n(1) = (-1)^n n! k^n}$$

A2. Either this is unusually trivial or I am missing something. Two of the points can be put on a great circle determining two hemispheres. One of these *closed* hemispheres must contain at least four of the points.

A3. We fix $n \geq 2$ (for a while) and if $\emptyset \neq S \subset 1, 2, \ldots, n$ we set

$$A(S) = \frac{1}{|S|} \sum_{k \in S}$$

where $|S|$ = cardinality of S. We have to prove: If

$$\mathcal{T}_n = \{S \subset 1, 2, \ldots, n \,:\, S \neq \emptyset, A(S) \text{ is an integer}\},$$

then $|\mathcal{T}_n| - n$ is even.

If is a subset of $\{1, \ldots, n\}$, let $S' = \{n - x + 1 \,:\, x \in S\}$ and observe that

$$A(s) + A(S') = n + 1$$

for all non-empty S. It is thus clear that the map $S \mapsto S'$,which preserves the cardinality of sets, is a bijection of $\mathcal{T}_n$ onto itself. We partition $\mathcal{T}_n = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A} = \{S \in \mathcal{T}_n \,:\, S = S'\}$, $B = \mathcal{T}_n \backslash \mathcal{A}$. Let $a = |\mathcal{A}|$, $b = |\mathcal{B}|$, so that $T_n = a + b$. It is clear that b is even since the map $S \mapsto S'$ is an involution without fixed points of $\mathcal{B}$ onto itself.

Assume now that n is even. Then $S = S'$ implies $2A(S) = A(S) + A(S') = n+1$, thus $A(S) = (n+1)/2$, which is not an integer. It follows that $\mathcal{A}$ is empty, and $T_n = b$ is even. On the other hand, if n is odd, the same calculation shows that *every* non-empty S such that $S' = S$ is in $\mathcal{A}$. A bit of reflection shows that $S = S' \neq \emptyset$ (for odd n) if and only if it has one of the two following forms:
i. $S = A \cup \{(n+1)/2\} \cup \{n - x + 1 \,:\, x \in A$ for some subset A of $\{1, \ldots, (n+1)/2\}$,
i. $S = A \cup \{n - x + 1 \,:\, x \in A$ for some *non-empty* subset A of $\{1, \ldots, (n+1)/2\}$.

It follows that

$$a = |\mathcal{A}| = 2^{(n+1)/2} + 2^{(n+1)/2} - 1$$

is odd, hence $T_n = a + b$ is also odd.

A4. I believe that I have a very involved proof that Player 0 always wins. If I am wrong, better not write it out. If right, it has too many cases.

A5. We prove by induction on the positive integer m: For every decomposition $m = a + b$ and a, b are positive integers relatively prime with respect to m, there exists a positive integer n such that $a_n = a$, $a_{n+1} = m$, and $a_{n+2} = b$. The case $m = 1$ is vacuously true, the case $m = 2$ is verified for $n = 2$. Assume the statement proved for all positive integers less than some integer m and assume $m = a + b$, where a, b are relatively prime with respect to m. Then a, b are also relatively prime with respect to each other. Assume first $a < b$. By the induction hypothesis, there exists k such that $a_k = a$, $a_{k+1} = b$ and $a_{k+2} = b - a$. It follows that $a_{2k+1} = a$, $a_{2k+2} = m$ and $a_{2k+3} = b$. If $b < a$, then the induction hypothesis provides k such that $a_k = a - b$, $a_{k+1} = a$, $a_{k+2} = b$. Then $a_{2k+3} = a$, $a_{2k+4} = m$, $a_{2k+5} = b$. The inductive proof is complete.

The result is now clear. If x is rational and $x < 1$, we can write $x = a/m$ with a, m relatively prime. There is then n with $a_n = a$, $a_{n+1} = m$ so that $x = a_n/a_{n+1}$. If $x > 1$, we use the fact that we can write $x = b/m$ with m, b relatively prime. Finally, of course, $1 = a_0/a_1$.

A6. The series converges if and only if $b = 2$.

For convenience we write

$$a_\ell = \sum_{k=b^{\ell-1}}^{b^\ell - 1} \frac{1}{k}$$

and we notice that

$$\log b \le a_\ell \le \log b + (b-1)b^{-\ell}$$

for $\ell = 1, 2, \ldots$, where $\log b$ is the natural logarithm of b.

Assume first $b > 2$. Let $n_1 = 1$ and $n_r = b^{n_{r-1}}$ if $r \ge 2$. Then, setting

$$S_r = \sum_{k=1}^{n_r - 1} \frac{1}{f(k)}$$

we see that

$$\begin{aligned} S_{r+1} &= \sum_{k=1}^{b^{n_r}-1} \frac{1}{f(k)} = \sum_{\ell=1}^{n_r} \sum_{k=b^{\ell-1}}^{b^\ell-1} \frac{1}{f(k)} \\ &= \sum_{\ell=1}^{n_r} \frac{1}{f(\ell)} a_\ell \ge \log b \sum_{\ell=1}^{n_r} \frac{1}{f(\ell)} \ge (\log b) S_r. \end{aligned}$$

By induction, $S_r \ge (\log b)^r S_1 = (\log b)^r$, proving the series diverges since $\log b > 1$ for $b > 2$; i.e., for $b \ge 3$.

Assume now that $b = 2$ and let n be a positive integer. Select the positive integer r so that $2^{r-1} \le n < 2^r$. Setting this time

$$S_n = \sum_{k=1}^{n} \frac{1}{f(k)},$$

we see that

$$\begin{aligned} S_n &\leq \sum_{k=1}^{2^r-1} \frac{1}{f(k)} = \sum_{\ell=1}^{r} \sum_{k=2^{\ell-1}}^{2^\ell-1} \frac{1}{f(k)} \\ &= \sum_{\ell=1}^{r} \frac{1}{f(\ell)} a_\ell \leq \sum_{\ell=1}^{r} \frac{1}{f(\ell)} (\log 2 + 2^{-\ell}). \end{aligned}$$

Since $r \leq n$ and $1/f(\ell) \leq 1$, we estimate

$$\sum_{\ell=1}^{r} \frac{1}{f(\ell)} (\log 2 + 2^{-\ell}) \leq (\log 2) S_n + \sum_{\ell=1}^{\infty} 2^{-\ell} = (\log 2) S_n + 2.$$

We proved $S_n \leq (\log 2) S_n + 2$; since $\log 2 < 1$ this proves $S_n \leq 2/(1 - \log 2)$ for all positive integers n. Convergence follows.