Putnam 2002

B1. Strangely enough, for any integer $n \geq 3$, any integer r, $1 \leq r \leq n-1$, the probability that S. O'N. (S from now on) will hit exactly r shots out of n is $1/(n-1)$. Thus, for $r = 50$, $n = 100$, the answer is

$$\frac{1}{99}.$$

Assume $n \geq 3$ and for $1 \leq r \leq n-1$, denote by $P(n,r)$ the probability that S hits r throws out of n. $P(n,1)$ is the probability of hitting exactly one throw, which can also be described as the probability of missing the n-th throw, given that only one throw was a hit in throws $1,\ldots,n-1$. We get

$$P(n,1) = P(n-1,1)(1 - \frac{1}{n-1}) = P(n-1,1)\frac{n-2}{n-1},$$

and since $P(3,1) = 1/2$, induction gives

$$P(n,1) = \frac{1}{2}\prod_{k=4}^{n}\frac{k-2}{k-1} = \frac{(n-2)!}{(n-1)!} = \frac{1}{n-1}.$$

Similarly, we see that

$$P(n,n-1) = \frac{1}{n-1}$$

for $n \geq 3$. In fact, $P(2,1) = 1$, hence $P(3,2) = 1/2$; $P(n,n-1)$ can be described as the probability of having a hit in every throw from the third one on, thus

$$P(n,n-1) = \prod_{k=3}^{n}\frac{k-2}{k-1} = \frac{(n-2)!}{(n-1)!} = \frac{1}{n-1}.$$

Assume proved that $P(n,r) = \frac{1}{n-1}$ for some $n \geq 3$, all r, $1 \leq r \leq n-1$. We proved this already for $n = 3$. In fact, if $n = 3$ the only values for r are $r = 1$ or $r = 2 = n-1$. To have the assertion proved for all n it thus suffices to prove it for n replaced by $n+1$. We may assume then that $2 \leq r \leq n-1$, since the cases $r = 1$, $r = n = (n+1)-1$ are done. Having r hits in $n+1$ throws breaks up into having r hits in n throws and then a miss in the (n+1)-st throw (probability $1-(r/n)$) or having $r-1$ hits inthe first n throws and then having another hit (probability $(r-1)/n$) in the $(n+1)$-st throw. Thus

$$P(n+1,r) = P(n,r)\frac{n-r}{n} + P(n,r-1)\frac{r-1}{n}.$$

By the induction hypothesis $P(n,r) = P(n,r-1) = 1/(n-1)$ so that

$$P(n+1,r) = \frac{n-r}{n(n-1)} + \frac{r-1}{n(n-1)} = \frac{1}{n}.$$

B2. We notice first that a polyhedron of the type described must have at least one face which is **not** a triangle. In fact, assume all faces are triangles and let V, F, E be the number of vertices, the number of faces, and the number of edges, respectively. Because every vertex has exactly three incident edges (and every edge has two vertices) we see that $E = 3V/2$. Now every face has three edges, every edge is common to two faces, so we also have $E = 3F/2$, thus $V = F$. By Euler's formula, $V + F - E \le 2$ (equal to 2 if the polyhedron has no holes), thus in this case we get

$$2F - \frac{3F}{2} \le 2, \qquad \text{hence } F \le 4.$$

Since the polyhedron has five or more faces, at least one of them, call it S, is not a triangle. The first player signs this face S. Suppose the second player signs a face that has an edge e in common with S. Since S is not a triangle, S has an edge f that has no common vertex with e and player 1 signs next a face having f as an edge. If player 2 signs a face that has no edge in common with S, then player 1 can sign any face with an edge common with S. Let S_2 be the second face signed by player 1, let v_1, v_2 be the vertices of the common edge f of S, S_1. If T is the third face of vertex v_1, U the third face of vertex v_2, then neither T nor U have been signed yet. The best player 2 can do on his second turn, is sign one of T or U, after which player 1 signs the remaining one and wins on his third turn.

B3. Let $g(x) = x - e\left(1 - \frac{1}{x}\right)^x x$ for $x \ge 1$. We claim that

$$\frac{1}{2} < g(x) < 1$$

for all real x, $x > 1$. Both inequalities can be proved by straightforward calculus arguments. To see $g(x) < 1$ for $x > 0$ we see that it is equivalent to proving

$$1 - \frac{1}{x} < e\left(1 - \frac{1}{x}\right)^x;$$

taking logarithms and replacing x by $z = 1/x$, the inequality reduces to proving that $z + (1-z)\log(1-z) > 0$ for $0 < z < 1$. Since the expression $z + (1-z)\log(1-z)$ equals 0 at 0 and its derivative with respectto z is $-\log(1-z) > 0$ in $(0,1)$, the inequality follows.

To see that $g(x) > 1/2$ is a bit more tedious. We first rewrite the inequality in the form

$$1 - \frac{1}{2x} > e\left(1 - \frac{1}{x}\right)^x;$$

we then take logarithms, replace x by $1/z$ to see that is is equivalent to

$$\phi(z) := z\log(1 - \frac{z}{2}) - \log(1-z) - z > 0$$

for $0 < z < 1$. We have

$$\phi'(z) = \log(1 - \frac{z}{2}) - \frac{z}{2-z} + \frac{1}{1-z} - 1,$$

$$\phi''(z) = \frac{z^3 - 5z^2 + 5z}{(2-z)^2(1-z)^2}.$$

We see that $\phi''(z) > 0$ for $0 < z < (5 - \sqrt{5})/2$, in particular in $(0, 1)$. Since $\phi'(0) = 0$, it follows that $\phi'(z) > 0$ in $(0, 1)$. Since $\phi(0) = 0$ it also follows that $\phi(z) > 0$ in $(0, 1)$, proving $g(x) > 1/2$ for all $x > 0$. The claim is established. Dividing by xe, it follows that

$$\frac{1}{2xe} < \frac{1}{e} - \left(1 - \frac{1}{x}\right)^x < \frac{1}{xe}$$

for all real numbers $x > 1$.