<u>Problem A1</u> Let n be a positive integer. How many ways are there to write n

as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots a_k \leq a_{k+1}$?

Solution There are n ways. It suffices to show that if k is an integer, $1 \leq k \leq n$, there is exactly one way to write n as a sum of k integers of the specified type. By the quotient algorithm, $n = kq + r$ for some r, $0 \leq r \leq k-1$. If $r = 0$, we can take $a_1 = a_2 = \cdots = a_k = q$ and write $n = a_1 + \cdots + a_k$. If $0 < r \leq k-1$, we can take $a_1 = \cdots = a_{k-r} = q$, $a_{k-r+1} = \cdots = a_k = q+1$ to get $a_1 + \cdots + a_k = (k-r)q + r(q+1) = n$. This shows there is at least one way to decompose n as specified for each k. To see this is the only such way, notice that if $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \leq a_2 \leq \cdots a_k \leq a_{k+1} \leq a_1 + 1$, then we either have $a_1 = a_2 = \cdots = a_k$ or there is ℓ, $1 < \ell < k$ such that $a_1 = a_2 = \cdots = a_\ell < a_{\ell+1} = \cdots = a_k = a_1 + 1$. In the former case $n = ka_1$, so that we are in the case $r = 0$, $q = a_1$, and the decomposition is the one given for that case. In the latter,

$$n = \ell a_1 + (k-\ell)(a_1+1) = ka_1 + (\ell - k)$$

so that we are in the case $r = k - \ell > 0$, and the decompostion coincides with the previous one. This proves the uniqueness of the decomposition for each k.