Problem A4 Suppose that a, b, c, A, B, C are real numbers , $a \neq 0$, $A \neq 0$, such that

$$\left|ax^2 + bx^2 + c\right| \leq \left|Ax^2 + Bx^2 + C\right|$$

for all real numbers x. Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$

Solution Dividing by x and letting $x \to \infty$, we see that $|a| \leq |A|$.

We consider three cases. **Case 1.** $B^2 - 4AC > 0$. Then the equation $Ax^2 + Bx + C = 0$ has two distinct roots. It follows that these must also be roots of the equation $ax^2 + bx + c = 0$, and the differences of the two roots of the first equation must coincide with the difference of the roots of the second. That is,

$$\frac{\sqrt{B^2 - 4AC}}{A} = \frac{\sqrt{b^2 - 4ac}}{a},$$

hence

$$\sqrt{B^2 - 4AC} = \frac{A}{a}\sqrt{b^2 - 4ac} \geq \sqrt{b^2 - 4ac},$$

and the desired inequality follows.

Case 2. $B^2 - 4AC = 0$ so that $Ax^2 + Bx + C = 0$ has a double root at $-B/2A$. The assumed inequality implies that

$$\limsup_{x \to A} \frac{|ax^2 + bx + c|}{\left(x + \frac{b}{2A}\right)^2} < \infty,$$

implying that $ax^2 + bx + c = 0$ also has a double root at $-B/2A$. Thus $-B/2A = -b/2a$ and in this case we have $b^2 - 4ac = 0 = B^2 - 4AC$.

Case 3. $B^2 - 4AC < 0$. The equation $Ax^2 + Bx + C = 0$ has no real roots. Assume $A > 0$, the case $A < 0$ is similar. In this case it is geometrically clear (and easy to prove analytically) that the y- coordinate of the vertex of the parabola $y = Ax^2 + Bx + C$ must be larger than the y coordinates of the vertices of the parabolas of equations $y = \pm(ax^2 + bx + c)$; i.e.,

$$\frac{4AC - B^2}{4A} \geq \frac{|4ac - b^2|}{4|a|},$$

hence

$$4AC - B^2 \geq \frac{A}{|a|}|4ac - b^2| \geq |4ac - b^2|.$$

The inequality has been proved in all cases.