Problem B2 Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$, by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries and continue until the final sequence consists of a single number x_n. Show that $x_n < 2/n$.

Solution By induction on n one sees that

$$x_n = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{k+1} = \frac{1}{2^{n-1}} \frac{2^n - 1}{n} < \frac{2}{n}.$$