<u>Problem B6</u> Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that

$$\int_0^1 \int_0^1 |f(x)+f(y)|\,dx\,dy \geq \int_0^1 |f(x)|\,dx.$$

Solution Let $A = \{x \in [0,1] : f(x) \geq 0\}$, $B = \{x \in [0,1] : f(x) < 0\}$ and let $|A|, |B|$ be the Lebesgue measures of A, B respectively. Then $0 \leq |A|, |B|$, $|A| + |B| = 1$. Setting

$$\begin{aligned} \mu &= \int_A f(x)\,dx = \int_A |f(x)|\,dx, \\ \nu &= -\int_B f(x)\,dx = \int_B |f(x)|\,dx, \end{aligned}$$

we get at once

$$\begin{aligned} \int_0^1 \int_0^1 |f(x)+f(y)|\,dx\,dy &= \int_{A\times A} (f(x)+f(y))\,dx\,dy \\ &+2\int_{A\times B} |f(x)-f(y)|\,dx\,dy - \int_{B\times B} (f(x)+f(y))\,dx\,dy \\ &\geq \int_{A\times A} (f(x)+f(y))\,dx\,dy + 2\left|\int_{A\times B} (f(x)-f(y))\,dx\,dy\right| \\ &- \int_{B\times B} (f(x)+f(y))\,dx\,dy = 2|A|\mu + 2|B|\nu + 2\left||A|\nu - |B|\mu\right|. \end{aligned}$$

We proved the following two inequalities

$$\begin{aligned} \int_0^1 \int_0^1 |f(x)+f(y)|\,dx\,dy &\geq 2(|A|-|B|)\mu + 2\nu, \\ \int_0^1 \int_0^1 |f(x)+f(y)|\,dx\,dy &\geq 2\mu + 2(|B|-|A|)\nu. \end{aligned}$$

Now $-1 \leq \gamma := |A| - |B| \leq 1$. If $\gamma \geq 0$, set $t = (1+2\gamma)/(2+2\gamma) \in (0,1)$. Multiplying the first of the two inequalities just mentioned by t, the second one by $1-t$ and adding yields

$$\begin{aligned} \int_0^1 \int_0^1 |f(x)+f(y)|\,dx\,dy &\geq 2\gamma t\mu + 2t\nu + 2(1-t)\mu - 2(1-t)\gamma\nu \\ &= \frac{2+2\gamma+4\gamma^2}{2+2\gamma}\mu + \nu \geq \mu + \nu = \int_0^1 |f(x)|\,dx. \end{aligned}$$

If $\gamma < 0$, we achieve the same result setting $t = (1-2\gamma)/(2-2\gamma)$, multiplying the first inequality by $1-t$, the second one by t, and adding.