Solution 1: We have \( \frac{A_1A_2}{BC} + \frac{B_1B_2}{BA} + \frac{C_1C_2}{CA} = 0 \), hence \( A_2B_1 + B_2C_1 + C_2A_1 = 0 \).

So, if \( A' \) or \( B' \) or \( C' \) is the common point of the lines \( A_2B_1 \) and \( C_2A_1 \) or \( B_2C_1 \) and \( A_2B_1 \), or \( C_2A_1 \) and \( B_2C_1 \), the triangle \( A'B'C' \) is equilateral.

The triangles \( AC_1B_2 \) and \( BA_1C_2 \) are congruent, because each of them is congruent with \( C'1C_2 \); the isometry mapping \( AC_1B_2 \) to \( BA_1C_2 \) is clearly the rotation with center \( O \) and angle \( 2\pi / 3 \), where \( O \) is the center of the triangle \( ABC \). Same way, the rotation \( (O, -2\pi / 3) \) maps the triangle \( AC_1B_2 \) to \( CB_1A_2 \).

Hence the triangle \( A_1B_1C_1 \) is equilateral with center \( O \) and the lines \( A_1B_2, B_1C_2 \), and \( C_1A_2 \) are the perpendicular bisectors of the sides of \( A_1B_1C_1 \); it follows that these three lines concur at \( O \).

Solution 2: If \( a_m = a_n \) with \( m < n \), we have \( a_m = a_n \) \( [n + 1] \) which is impossible. Hence, the \( a_n \) for \( n \in \mathbb{Z} \), are distinct.

If \( a_n = \min(a_1, a_2, ..., a_n) \) and \( \beta_n = \max(a_1, a_2, ..., a_n) \), as \( a_n = \beta_n - a_n \), we have \( \beta_n - a_n < n \) and, \( a_1, a_2, ..., a_n \) being distinct, \{\( a_1, a_2, ..., a_n \}\} is a set of \( n \) consecutive integers.

As \{\( a_n \mid n \in \mathbb{Z} \}\} has infinitely many positive and negative elements, it follows that \( \{a_n \mid n \in \mathbb{Z}\} = \mathbb{Z} \).

Solution 3: We have

\[
\left[(x^2 + y^2 + z^2)x^3 - (x^3 + y^2 + z^2)\right] \left(x^2 - \frac{1}{x}\right) = \frac{(y^2 + z^2)(x^3 - 1)^2}{x} \geq 0
\]

Hence \( \frac{x^5 - x^2}{x^2 + y^2 + z^2} \geq \frac{x^2 - \frac{1}{x}}{x^2 + y^2 + z^2} = \frac{x^2 - yz}{x^2 + y^2 + z^2} \).

As \( (x^2 - yz) + (y^2 - zx) + (z^2 - xy) \geq 0 \), the result follows.

Solution 4: If \( p \) is prime \( > 3 \), we have

\( 6, a_{p-2} = 3(2^p - 1) + 2(3^p - 1) + (6^p - 1) \) and, by Fermat’s theorem, \( p \) divides \( a_{p-2} \).

As \( 2 \) and \( 3 \) divide \( a_2 = 48 \), it follows that every prime \( p \) divides one of the \( a_n \); hence \( 1 \) is the only positive integer relatively prime to all the \( a_n \).

Solution 5: The common point \( \Omega \) of the perpendicular bisectors of \( [AC] \) and \( [BD] \) is the center of the rotation mapping \( C \) to \( A \), \( B \) to \( D \) and \( E \) to \( F \). Thus \( \Omega \) is the second intersection (apart \( D \)) of the circles \( DAP \) and \( DFQ \). So, if \( <d, d'\) is the directed angle of the lines \( d \) and \( d' \), we have (modulo \( \pi \))

\( <RP, RQ =< AP, FQ =< AP, AD =< FD, FQ =< \Omega P, \Omega D =< \Omega D, \Omega Q =< \Omega P, \Omega Q \) and the circle \( PQR \) goes through \( \Omega \) (this is also a consequence of Miquel’s theorem).

Solution 6: Let \( f(i, j) \) if \( i \neq j \), be the number of contestants who have solved the problems \( i \) and \( j \); let \( n \) be the number of contestants.

As \( f(i, j) \geq \frac{2n + 1}{5} \), we have \( S = \sum_{1 \leq i < j \leq 6} f(i, j) \geq 15 \left( \frac{2n + 1}{5} \right) = 6n + 3 \).

If each contestant has solved at most 4 problems, we have \( \sum_{1 \leq i < j \leq 6} f(i, j) \leq nC_4 = 6n \), which is impossible.

Suppose now that only one contestant has solved 5 problems; say, the problems 1, 2, 3, 4, 5. If \( p \) is the number of contestants who have solved exactly 4 problems, we have \( 6n + 3 \leq S \leq C_3^2 + pC_4^2 + (n - p - 1)C_3^3 \), hence \( p \geq n - \frac{1}{3} \) and \( p = n - 1 \).

So \( S = 10 + 6(n - 1) = 6n + 4 \). Moreover, \( k = \frac{2n + 1}{5} \in \mathbb{N} \) because, elsewhere,

\( S \geq 15 \left( \frac{2n + 2}{5} \right) = 6n + 6 \).
From \( \sum_{1 \leq i, j \leq 6} [f(i,j) - k] = 1 \), it follows that each \( f(i,j) \), for \( i < j \), is \( k \) except for one of them, \( f(i_0, j_0) \) whose value is \( k + 1 \).

If \( \varphi(t) = \sum_{i \leq t} f(i,t) \), then \( \varphi(6) \) is 3 times the number of contestants who have solved the problem 6 and, for \( 1 \leq t < 6 \), \( \varphi(t) - 1 \) is 3 times the number of contestants who have solved the problem \( t \).

Hence, for \( 1 \leq t < 6 \), we have \( \varphi(t) \neq \varphi(6) \).

But, if \( t \not\in \{i_0, j_0, 6\} \), we have \( \varphi(t) = \varphi(6) = \begin{cases} 5k & \text{si } j_0 < 6 \\ 5k + 1 & \text{si } j_0 = 6 \end{cases} \) and the contradiction.