Friendship Among Triangle Centers

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Abstract. If we erect on the sides of a scalene triangle three squares, then at the vertices of the triangle we find new triangles, the flanks. We study pairs of triangle centers \( X \) and \( Y \) such that the triangle of \( X \)'s in the three flanks is perspective with \( ABC \) at \( Y \), and vice versa. These centers \( X \) and \( Y \) we call friends. Some examples of friendship among triangle centers are given.

1. Flanks

Given a triangle \( ABC \) with side lengths \( BC = a \), \( CA = b \), and \( AB = c \). By erecting squares \( AC_aC_bB \), \( BA_bA_cC \), and \( CB_cB_aA \) externally on the sides, we form new triangles \( AB_aC_a \), \( BC_bA_b \), and \( CA_cB_c \), which we call the flanks of \( ABC \). See Figure 1.

![Figure 1](image1.png)

If we rotate the \( A \)-flank (triangle \( AB_aC_a \)) by \( \frac{\pi}{2} \) about \( A \), then the image of \( C_a \) is \( B \), and that of \( B_a \) is on the line \( CA \). Triangle \( ABC \) and the image of the \( A \)-flank form a larger triangle in which \( BA \) is a median. From this, \( ABC \) and the \( A \)-flank have equal areas. It is also clear that \( ABC \) is the \( A \)-flank triangle of the \( A \)-flank triangle. These observations suggest that there are a close relationship between \( ABC \) and its flanks.

2. Circumcenters of flanks

If \( P \) is a triangle center of \( ABC \), we denote by \( P_A \), \( P_B \), and \( P_C \) the same center of the \( A \)-, \( B \)-, and \( C \)- flanks respectively.

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Let \( O \) be the circumcenter of triangle \( ABC \). Consider the triangle \( O_AO_BO_C \) formed by the circumcenters of the flanks. By the fact that the circumcenter is the intersection of the perpendicular bisectors of the sides, we see that \( O_AO_BO_C \) is homothetic (parallel) to \( ABC \), and that it bisects the squares on the sides of \( ABC \). The distances between the corresponding sides of \( ABC \) and \( O_AO_BO_C \) are therefore \( \frac{a}{2}, \frac{b}{2}, \) and \( \frac{c}{2} \).

3. Friendship of circumcenter and symmedian point

Now, homothetic triangles are perspective at their center of similitude. The distances from the center of similitude of \( ABC \) and \( O_AO_BO_C \) to the sides of \( ABC \) are proportional to the distances between the corresponding sides of the two triangles, and therefore to the sides of \( ABC \). This perspector must be the symmedian point \( K \).

The triangle \( O_AO_BO_C \) of circumcenters of the flanks is perspective with \( ABC \) at the symmedian point \( K \) of \( ABC \). In particular, the \( A \)-Cevian of \( K \) in \( ABC \) (the line \( AK \)) is the same line as the \( A \)-Cevian of \( O_A \) in the \( A \)-flank. Since \( ABC \) is the \( A \)-flank of triangle \( AB_OC \), the \( A \)-Cevian of \( K_A \) in the \( A \)-flank is the same line as the \( A \)-Cevian of \( O \) in \( ABC \) as well. Clearly, the same statement can be made for the \( B \)- and \( C \)-flanks. The triangle \( K_AK_BK_C \) of symmedian points of the flanks is perspective with \( ABC \) at the circumcenter \( O \).

For this relation we call the triangle centers \( O \) and \( K \) friends. See Figure 3. More generally, we say that \( P \) befriends \( Q \) if the triangle \( PA_PB_PC \) is perspective with \( ABC \) at \( Q \). Such a friendship relation is always symmetric since, as we have remarked earlier, \( ABC \) is the \( A \)-, \( B \)-, \( C \)-flank respectively of its \( A \)-, \( B \)-, \( C \)-flanks.

\[1\] This is \( X_6 \) in [2, 3].
4. Isogonal conjugacy

It is easy to see that the bisector of an angle of \( ABC \) also bisects the corresponding angle of its flank. The incenter of a triangle, therefore, befriends itself.

Consider two friends \( P \) and \( Q \). By reflection in the bisector of angle \( A \), the line \( PAQ_A \) is mapped to the line joining the isogonal conjugates of \( P \) and \( Q_A \).\(^2\) We conclude:

**Proposition.** If two triangle centers are friends, then so are their isogonal conjugates.

Since the centroid \( G \) and the orthocenter \( H \) are respectively the isogonal conjugates of the symmedian point \( K \) and the circumcenter \( O \), we conclude that \( G \) and \( H \) are friends.

5. The Vecten points

The centers of the three squares \( AC_aC_bB \), \( BA_bA_C \) and \( CB_cB_aA \) form a triangle perspective with \( ABC \). The perspector is called the *Vecten point* of the triangle.\(^3\) By the same token the centers of three squares constructed *inwardly* on the three sides also form a triangle perspective with \( ABC \). The perspector is called the *second Vecten point*.\(^4\) We show that each of the Vecten points befriends itself.

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\(^2\)For \( Q_A \), this is the same line when isogonal conjugation is considered both in triangle \( ABC \) and in the \( A \)-flank.

\(^3\)This is the point \( X_{485} \) of [3].

\(^4\)This is the point \( X_{486} \) of [3], also called the *inner* Vecten point.
6. The Second Vecten points

O. Bottema [1] has noted that the position of the midpoint \( M \) of segment \( BC_b \) depends only on \( B, C \), but not on \( A \). More specifically, \( M \) is the apex of the isosceles right triangle on \( BC \) pointed towards \( A \).\(^5\)

To see this, let \( A', M', B'_c \) and \( C'_b \) be the orthogonal projections of \( A, M, B_c \) and \( C_b \) respectively on the line \( BC \). See Figure 4. Triangles \( AA'C \) and \( CB_b'C_b \) are congruent by rotation through \( \pm \frac{\pi}{2} \) about the center of the square \( CB_b'B_c \). Triangles \( AA'B \) and \( BC_b'B_c \) are congruent in a similar way. So we have \( AA' = CB_b' = BC_b = BC_c' \). It follows that \( M' \) is also the midpoint of \( BC \). And we see that \( C'_b + B'_c + B_c = BA' + A'C = a \) so \( MM' = \frac{a}{2} \). And \( M \) is as desired.

By symmetry \( M \) is also the apex of the isosceles right triangle on \( B_aC_a \) pointed towards \( A \).

We recall that the triangle of apexes of similar isosceles triangles on the sides of \( ABC \) is perspective with \( ABC \). The triangle of apexes is called a Kiepert triangle, and the Kiepert perspector \( K(\phi) \) depends on the base angle \( \phi \) (mod \( \pi \)) of the isosceles triangle.\(^6\)

We conclude that \( AM \) is the \( A \)-Cevian of \( K(-\frac{\pi}{4}) \), also called the second Vecten point of both \( ABC \) and the \( A \)-flank. From similar observations on the \( B \)- and \( C \)-flanks, we conclude that the second Vecten point befriends itself.

7. Friendship of Kiepert perspectors

Given any real number \( t \), let \( X_t \) and \( Y_t \) be the points that divide \( CB_c \) and \( BC_b \) such that \( CX_t : CB_c = BY_t : BC_b = t : 1 \), and let \( M_t \) be their midpoint. Then \( BCM_t \) is an isosceles triangle, with base angle \( \arctan t = \angle BAY_t \). See Figure 5.

Extend \( AX_t \) to \( X'_t \) on \( B_aB_c \), and \( AY_t \) to \( Y'_t \) on \( C_aC_b \) and let \( M'_t \) be the midpoint of \( X'_tY'_t \). Then \( B_aC_aM'_t \) is an isosceles triangle, with base angle \( \arctan \frac{1}{t} = \angle Y'_tAC_a = \frac{\pi}{2} - \angle BAY_t \). Also, by the similarity of triangles \( AX_tY_t \) and \( AX'_tY'_t \)

\(^5\)Bottema introduced this result with the following story. Someone had found a treasure and hidden it in a complicated way to keep it secret. He found three marked trees, \( A, B \) and \( C \), and thought of rotating \( BA \) through 90 degrees to \( BC_b \), and \( CA \) through \(-90 \) degrees to \( CB_c \). Then he chose the midpoint \( M \) of \( C_bB_c \) as the place to hide his treasure. But when he returned, he could not find tree \( A \). He decided to guess its position and try. In a desperate mood he imagined numerous
we see that $A$, $M_t$ and $M'_t$ are collinear. This shows that the Kiepert perspectors $K(\phi)$ and $K(\frac{\pi}{2} - \phi)$ are friends.

In particular, the first Vecten point $K(\frac{\pi}{4})$ also befriends itself. See Figure 6. The Fermat points $K(\pm \frac{\pi}{3})$ are friends of the Napoleon points $K(\frac{\pi}{6})$.  

Seen collectively, the Kiepert hyperbola, the locus of Kiepert perspectors, befriends itself; so does its isogonal transform, the Brocard axis $OK$.

diggings without result. But, much to his surprise, he was able to recover his treasure on the very first try!

\footnote{By convention, $\phi$ is positive or negative according as the isosceles triangles are pointing outwardly or inwardly.}

\footnote{These are the points $X_{13}$ and $X_{14}$ in [2, 3], also called the isogenic centers.}

\footnote{These points are labelled $X_{17}$ and $X_{18}$ in [2, 3]. It is well known that the Kiepert triangles are equilateral.}
References


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