Euler’s Formula and Poncelet’s Porism

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1. Introduction

It is well known [2, p. 187] that two intersecting circles \( O(R) \) and \( O_1(R_1) \) are the circumcircle and an excircle respectively of a triangle if and only if the Euler formula

\[
d^2 = R^2 + 2RR_1,
\]

where \( d = |OO_1| \), holds. We present a possibly new proof and an application to the Poncelet porism.

![Figure 1](image1.png)

**Theorem 1.** Intersecting circles \( (O) \) and \( (O_1) \) are the circumcircle and an excircle of a triangle if and only if the tangent to \( (O_1) \) at an intersection of the circles meets \( (O) \) again at the touch point of a common tangent.

**Proof.** (Sufficiency) Let \( O(R) \) and \( O_1(R_1) \) be intersecting circles. (These circles are not assumed to be related to a triangle as in Figure 1.) Of the two lines tangent to both circles, let \( AK \) be one of them, as in Figure 2. Let \( P = AK \cap OO_1 \). Of the two points of intersection of \( (O) \) and \( (O_1) \), let \( C \) be the one not on the same side of line \( OO_1 \) as point \( A \). Line \( AC \) is tangent to circle \( O_1(R_1) \) if and only if \( |AC| = |AK| \). Let \( B \) and \( M \) be the points other than \( C \) where line \( PC \) meets circles \( O(R) \) and \( O_1(R_1) \), respectively. Triangles \( ABC \) and \( KCM \) are homothetic with ratio \( \frac{R}{R_1} \), so that

\[
\frac{|AB|}{|CK|} = \frac{R}{R_1}.
\]

Also, triangles \( ABC \) and \( CAK \) are similar, since \( \angle ABC = \angle CAK \) and \( \angle BAC = \angle ACK \). Therefore,

\[
\frac{|AB|}{|AC|} = \frac{|AC|}{|CK|},
\]

so that

\[
\frac{|CK|}{|AC|} \cdot \frac{R}{R_1} = \frac{|AC|}{|CK|},
\]

and

\[
|CK| = |AC| \sqrt{\frac{R_1}{R}}.
\]

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Also,

\[ |AK| = |AC| \cos(\angle CAK) + |CK| \cos(\angle CKA) \]
\[ = |AC| \sqrt{1 - \frac{|AC|^2}{4R^2}} + |CK| \sqrt{1 - \frac{|CK|^2}{4R_1^2}}. \quad (3) \]

If \( |AC| = |AK| \), then equations (2) and (3) imply

\[ |AK| = |AK| \sqrt{1 - \frac{|AK|^2}{4R^2}} + |AK| \sqrt{\frac{R_1}{R} - \frac{|AK|^2}{4R^2}}, \]

which simplifies to \( |AK|^2 = 4RR_1 - R_1^2 \). Since \( |AK|^2 = d^2 - (R - R_1)^2 \), where \( d = |OO_1| \), we have the Euler formula given in (1).

We shall prove the converse below from Poncelet’s porism.

2. Poncelet porism

Suppose triangle \( ABC \) has circumcircle \( O(R) \) and incircle \( I(r) \). The Poncelet porism is the problem of finding all triangles having the same circumcircle and incircle, and the well known solution is an infinite family of triangles. Unless triangle \( ABC \) is equilateral, these triangles vary in shape, but even so, they may be regarded as “rotating” about a fixed incircle and within a fixed circumcircle.

Continuing with the proof of the necessity part of Theorem 1, let \( I_1(r_1) \) be the excircle corresponding to vertex \( A \). Since Euler’s formula holds for this configuration, the conditions for the Poncelet porism (e.g. [2, pp. 187-188]) hold. In the family of rotating triangles \( ABC \) there is one whose vertices \( A \) and \( B \) coincide in a point, \( D \), and the limiting line \( AB \) is, in this case, tangent to the excircle. Moreover, lines \( CA \) and \( BC \) coincide as the line tangent to the excircle at a point of intersection of the circles, as in Figure 3. This completes the proof of Theorem 1.

Certain points of triangle \( ABC \), other than the centers of the two fixed circles, stay fixed during rotation ([1, p.16-19]). We can also find a fixed line in the Poncelet porism.
Theorem 2. For each of the rotating triangles $ABC$ with fixed circumcircle and excircle corresponding to vertex $A$, the feet of bisectors $BB_1$ and $CC_1$ traverse line $DE$, where $E$ is the touch point of the second common tangent.

3. Proof of Theorem 2

We begin with the pole-polar correspondence between points and lines for the excircle with center $I_1$, as in Figure 4.

The polars of $A, B, C$ are $LM, MK, KL$, respectively, where $\triangle KLM$ is the $A$-extouch triangle. As $BB_1$ is the internal bisector of angle $B$ and $BI_1$ is the external bisector, we have $BB_1 \perp BI_1$, and the pole of $BB_1$ lies on the polar of $B$, namely $MK$. Therefore the pole of $BB_1$ is the midpoint $P$ of segment $MK$. Similarly, the pole of the bisector $CC_1$ is the midpoint $Q$ of segment $KL$. The polar of $B_1$ is the line passing through the poles of $BB_1$ and $LB_1$, i.e. line $PL$. Likewise, $MQ$ is the polar of $C_1$, and the pole of $B_1C_1$ is centroid of triangle $KLM$, which we denote as $G_1$.

We shall prove that $G_1$ is fixed by proving that the orthocenter $H_1$ of triangle $KLM$ is fixed. (Gallatly [1] proves that the orthocenter of the intouch triangle stays fixed in the Poncelet porism with fixed circumcircle and incircle; we offer a different proof, which applies also to the circumcircle and an excircle.)

Lemma 3. The orthocenter $H_1$ of triangle $KLM$ stays fixed as triangle $ABC$ rotates.

Proof. Let $KLM$ be the extouch triangle of triangle $ABC$, let $RST$ be the orthic triangle of triangle $KLM$, and let $H_1$ and $E_1$ be the orthocenter and nine-point center, respectively, of triangle $KLM$, as in Figure 5.

1) The circumcircle of triangle $RST$ is the nine-point circle of triangle $KLM$, so that its radius is equal to $\frac{1}{2}R_1$, and its center $E_1$ is on the Euler line $I_1H_1$ of triangle $KLM$.

2) It is known that altitudes of an obtuse triangle are bisectors (one internal and two external) of its orthic triangle, so that $H_1$ is the $R$-excenter of triangle $RST$. 
(3) Triangle $RST$ and triangle $ABC$ are homothetic. To see, for example, that $\overline{AB} \parallel \overline{RS}$, we have $\angle KRL = \angle KSL = 90^\circ$, so that $L, R, S, K$ are concyclic. Thus, $\angle KLR = \angle KSR = \angle RSM$. On the other hand, $\angle KLR = \angle KLM = \angle KMB$ and $\angle RSM = \angle SMB$. Consequently, $\overline{AB} \parallel \overline{RS}$.

(4) The ratio $k$ of homothety of triangle $ABC$ and triangle $RST$ is equal to the ratio of their circumradii, i.e. $k = \frac{2R}{R_1}$. Under this homothety, $O \rightarrow E_1$ (the circumcenters) and $I_1 \rightarrow H_1$ (the excenter). It follows that $OI_1 || E_1 H_1$. Since $E_1$, $I_1$, $H_1$ are collinear, $O$, $I_1$, $H_1$ are collinear. Thus $OI_1$ is the fixed Euler line of every triangle $KLM$.

The place of $H$ stays fixed on $OI$ because $EH = \frac{OI}{k}$ remains constant. Therefore the centroid of, triangle $KLM$ also stays fixed. $\square$

To complete the proof of Theorem 2, note that by Lemma 3, $G_1$ is fixed on line $OI_1$. Therefore, line $B_1C_1$, as the polar of $G_1$, is fixed. Moreover, $B_1C_1 \perp OI_1$. Considering the degenerate case of the Poncelet porism, we conclude that $B_1C_1$ coincides with $DE$, as in Figure 3.

References


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