Congruent Inscribed Rectangles

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Abstract. We solve the construction problem of an interior point $P$ in a given triangle $ABC$ with congruent rectangles inscribed in the subtriangles $PBC$, $PCA$ and $PAB$.

1. Congruent inscribed rectangles

Given a triangle with sidelengths $a, b, c$, let $L_m = \min (a, b, c)$; $L \in (0, L_m)$ and $\mu > 0$. Let $P$ be a point inside $ABC$ with distances $d_a, d_b, d_c$ to the sidelines of $ABC$. Suppose that a rectangle with lengths of sides $L$ and $\mu L$ is inscribed in the triangle $PBC$, with two vertices with distance $L$ on the segment $BC$, the other vertices on the segments $PB$ and $PC$. Then, $\frac{L}{d_a - \mu L} = \frac{a}{d_a}$, or $d_a = \frac{\mu a L}{a - L}$.

If we can inscribe congruent rectangles with side lengths $L$ and $\mu L$ in the three triangles $PBC$, $PCA$, $PAB$, we have necessarily

$$f_\mu(L) := \frac{a^2}{a - L} + \frac{b^2}{b - L} + \frac{c^2}{c - L} - \frac{2\Delta}{\mu L} = 0,$$

where $\Delta$ is the area of triangle $ABC$. This is because $ad_a + bd_b + cd_c = 2\Delta$.

The function $f_{\mu}(L)$ increases from $-\infty$ to $+\infty$ when $L$ moves on $(0, L_m)$. The equation $f_{\mu}(L) = 0$ has a unique root $L_{\mu}$ in $(0, L_m)$ and the point

$$P_{\mu} = \left( \frac{a^2}{a-L_{\mu}} : \frac{b^2}{b-L_{\mu}} : \frac{c^2}{c-L_{\mu}} \right)$$

in homogeneous barycentric coordinates is the only point $P$ inside $ABC$ for which we can inscribe congruent rectangles with side lengths $L_{\mu}$ and $\mu L_{\mu}$ in the three triangles $PBC, PCA, PAB$. If $\mathcal{H}_0$ is the circumhyperbola through $I$ (incenter) and $K$ (symmedian point), the locus of $P_{\mu}$ when $\mu$ moves on $(0, +\infty)$ is the open arc $\Omega$ of $\mathcal{H}_0$ from $I$ to the vertex of $ABC$ opposite to the shortest side. See Figure 1. For $\mu = 1$, the smallest root $L_1$ of $f_1(L) = 0$ leads to the point $P_1$ with congruent inscribed squares.

![Figure 2](image)

2. Construction of congruent inscribed rectangles

Consider $P \in \Omega$, $Q$ and $E$ the reflections of $P$ and $C$ with respect to the line $IB$. The parallel to $AB$ through $Q$ intersects $BP$ at $F$. The lines $EF$ and $AP$ intersect at $X$. Then the parallel to $AB$ through $X$ is a sideline of the rectangle inscribed in $PAB$. The reflections of this line with respect to $AI$ and $BI$ will each give a sideline of the two other rectangles.\(^1\)

**Proof.** We have $\frac{BE}{BA} = \frac{a}{c}$, $\frac{BF}{BP} = \frac{d_c}{d_a} = \frac{c}{a} \frac{a - L_{\mu}}{c - L_{\mu}}$. Applying the Menelaus theorem to triangle $PAB$ and transversal $EFX$, we have

$$\frac{XA}{XP} = \frac{FB}{FP} \frac{EA}{EB} = \frac{L_{\mu} - c}{L_{\mu}}.$$

More over, the sidelines of the rectangles parallel to $BC, CA, AB$ form a triangle homothetic at $I$ with $ABC$. \(\square\)

\(^1\)This construction was given by Bernard Gibert.
3. Construction of $P_\mu$

The point $P_\mu$ is in general not constructible with ruler and compass. We give here a construction as the intersection of the arc $\Omega$ with a circle.

Consider the points

$$X_{100} = \left( \frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b} \right)$$

and

$$X_{106} = \left( \frac{a^2}{b+c-2a} : \frac{b^2}{c+a-2b} : \frac{c^2}{a+b-2c} \right)$$

on the circumcircle. Note that the line $X_{100}X_{106}$ passes through the incenter $I$. The line joining $X_{106}$ to the symmedian point $K$ intersects the circumcircle again at

$$X_{101} = \left( \frac{a^2}{b-c} : \frac{b^2}{c-a} : \frac{c^2}{a-b} \right).$$

Construction. Draw outwardly a line $\ell$ parallel to $AC$ at a distance $\mu b$ from $AC$, intersecting the line $CK$ at $S$. The parallel at $S$ to the line $CX_{101}$ intersects the line $KX_{101}X_{106}$ at $Y_\mu$. Then $P_\mu$ is the intersection of the arc $\Omega$ with the circle through $X_{100}$, $X_{106}$, and $Y_\mu$. See Figure 4.

Proof. From

$$L = \frac{2a\Delta x}{2\Delta x + \mu a^2(x+y+z)} = \frac{2b\Delta y}{2\Delta x + \mu b^2(x+y+z)} = \frac{2c\Delta z}{2\Delta z + \mu c^2(x+y+z)},$$

\footnote{We follow the notations of [1]. Here, $X_{100}$ is the isogonal conjugate of the infinite point of the trilinear polar of the incenter, and $X_{106}$ is the isogonal conjugate of the infinite point of the line $GI$ joining the centroid and the incenter.}
we note that \( P_\mu \) lies on the three hyperbolas \( \mathcal{H}_a, \mathcal{H}_b, \) and \( \mathcal{H}_c \) with equations

\[
\begin{align*}
\mu bc(x + y + z)(cy - bz) + 2\Delta(b - c)yz &= 0, \\
\mu ca(x + y + z)(az - cx) + 2\Delta(c - a)zx &= 0, \\
\mu ab(x + y + z)(bx - ay) + 2\Delta(a - b)xy &= 0.
\end{align*}
\]

Computing \( a^2(b - c)(c - a)\mathcal{H}_a + b^2(b - c)(a - b)\mathcal{H}_b + c^2(c - a)(b - c)\mathcal{H}_c \), we see that \( P_\mu \) lies on the circle \( \Gamma_\mu \):

\[
\mu abc(x + y + z)\Lambda + 2\Delta(a - b)(b - c)(c - a)(a^2yz + b^2zx + c^2xy) = 0,
\]

where

\[
\Lambda = bc(b - c)(b + c - 2a)x + ca(c - a)(c + a - 2b)y + ab(a + b - 2c)(a - b)z.
\]

As \( \Lambda = 0 \) is the line \( X_{100}X_{106} \), the circle \( \Gamma_\mu \) passes through \( X_{100} \) and \( X_{106} \).

Now, as \( \ell \) is the line \( 2\Delta y + \mu b^2(x + y + z) = 0 \), we have

\[
S = \left( a^2 : b^2 : - \left( a^2 + b^2 + \frac{2\Delta}{\mu} \right) \right).
\]

The parallel through \( S \) to \( CX_{101} \) is the line

\[
\mu(b + a - 2c)(x + y + z) + 2\Delta \left( \frac{(b - c)x}{a^2} + \frac{(a - c)y}{b^2} \right) = 0,
\]

and \( KX_{101} \) is the line

\[
b^2c^2(b - c)(b + c - 2a)x + c^2a^2(c - a)(c + a - 2b)y + a^2b^2(a - b)(a + b - 2c)z = 0.
\]

We can check that these two lines intersect at the point

\[
Y_\mu = (a^2(2\Delta(c - a)(a - b) + \mu(-a^2b^2 + c^4) + 2abc(b + c) + (b^4 - 2b^3c - 2b^3c + c^4))

+ b^2(2\Delta(a - b)(b - c) + \mu(-b^2(c^2 + a^2) + 2abc(c + a) + (c^4 - 2c^3a - 2ca^3 + a^4))

+ c^2(2\Delta(b - c)(c - a) + \mu(-c^2(a^2 + b^2) + 2abc(a + b) + (a^4 - 2a^3b - 2ab^3 + b^4)))
\]

on the circle \( \Gamma_\mu \).

Remark. The circle through \( X_{100}, \) \( X_{106} \) and \( P_\mu \) is the only constructible circle through \( P_\mu \), and there is no constructible line through \( P_\mu \).

References

http://cedar.evansville.edu/~ck6/encyclopedia/.

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