The Lemoine Cubic and Its Generalizations

Bernard Gibert

Abstract. For a given triangle, the Lemoine cubic is the locus of points whose cevian lines intersect the perpendicular bisectors of the corresponding sides of the triangle in three collinear points. We give some interesting geometric properties of the Lemoine cubic, and study a number of cubics related to it.

1. The Lemoine cubic and its constructions

In 1891, Lemoine published a note [5] in which he very briefly studied a cubic curve defined as follows. Let $M$ be a point in the plane of triangle $ABC$. Denote by $M_a$ the intersection of the line $MA$ with the perpendicular bisector of $BC$ and define $M_b$ and $M_c$ similarly. The locus of $M$ such that the three points $M_a$, $M_b$, $M_c$ are collinear on a line $L_M$ is the cubic in question. We shall denote this cubic by $K(O)$, and follow Neuberg [8] in referring to it as the Lemoine cubic.

Lemoine claimed that the circumcenter $O$ of the reference triangle was a triple point of $K(O)$. As pointed out in [7], this statement is false. The present paper considerably develops and generalizes Lemoine’s note.

We use homogeneous barycentric coordinates, and adopt the notations of [4] for triangle centers. Since the second and third coordinates can be obtained from the first by cyclic permutations of $a$, $b$, $c$, we shall simply give the first coordinates. For convenience, we shall also write

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}.$$ 

Thus, for example, the circumcenter is $X_3 = [a^2 S_A]$.

Figure 1 shows the Lemoine cubic $K(O)$ passing through $A$, $B$, $C$, the orthocenter $H$, the midpoints $A'$, $B'$, $C'$ of the sides of triangle $ABC$, the circumcenter $O$, and several other triangle centers such as $X_{52} = [a^4]$, $X_{56} = \left[\frac{a^2}{b+c-a}\right]$ and its extraversions. 1 Contrary to Lemoine’s claim, the circumcenter is a node. When $M$ traverses the cubic, the line $L_M$ envelopes the Kiepert parabola with focus

1The three extraversions of a point are each formed by changing in its homogeneous barycentric coordinates the signs of one of $a$, $b$, $c$. Thus, $X_{56a} = \left(\frac{a^2}{b+c+a} : \frac{b^2}{c+a-b} : \frac{c^2}{a+b-c}\right)$, and similarly for $X_{56b}$ and $X_{56c}$. 


The author sincerely thanks Edward Brisse, Jean-Pierre Ehrmann and Paul Yiu for their friendly and efficient help. Without them, this paper would never have been the same.
\( F = X_{110} = \left[ \frac{a^2}{b^2 - c^2} \right] \) and directrix the Euler line. The equation of the Lemoine cubic is

\[
\sum_{\text{cyclic}} a^4 S_{Ayz}(y - z) + (a^2 - b^2)(b^2 - c^2)(c^2 - a^2)xyz = 0.
\]

Figure 1. The Lemoine cubic with the Kiepert parabola

We give two equivalent constructions of the Lemoine cubic.

**Construction 1.** For any point \( Q \) on the line \( GK \), the trilinear polar \( q \) of \( Q \) meets the perpendicular bisectors \( OA', OB', OC' \) at \( Q_a, Q_b, Q_c \) respectively. \(^2\) The lines \( AQ_a, BQ_b, CQ_c \) concur at \( M \) on the cubic \( \mathcal{K}(O) \).

For \( Q = (a^2 + t : b^2 + t : c^2 + t) \), this point of concurrency is

\[
M = \left( \frac{a^2 + t}{b^2 + t} : \frac{b^2 + t}{c^2 + t} : \frac{c^2 + t}{a^2 + t} \right).
\]

\(^2\)The tripolar \( q \) envelopes the Kiepert parabola.
This gives a parametrization of the Lemoine cubic. This construction also yields the following points on \( K(O) \), all with very simple coordinates, and are not in [4].

<table>
<thead>
<tr>
<th>( i )</th>
<th>( Q = X_i )</th>
<th>( M = M_i )</th>
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<tbody>
<tr>
<td>69</td>
<td>( S_A )</td>
<td>( \frac{S_A}{a^2+c^2-a^4} )</td>
</tr>
<tr>
<td>86</td>
<td>( \frac{1}{b+c} )</td>
<td>( \frac{1}{a(b+c)-(b^2+bc+c^2)} )</td>
</tr>
<tr>
<td>141</td>
<td>( b^2 + c^2 )</td>
<td>( \frac{b^2+c^2+c^2-a^4}{a(b+c)-(b^2+bc+c^2)} )</td>
</tr>
<tr>
<td>193</td>
<td>( b^2 + c^2 - 3a^2 )</td>
<td>( S_A(b^2 + c^2 - 3a^2) )</td>
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</table>
Construction 2. For any point \( Q \) on the Euler line, the perpendicular bisector of \( FQ \) intersects the perpendicular bisectors \( OA', OB', OC' \) at \( Q_a, Q_b, Q_c \) respectively. The lines \( AQ_a, BQ_b, CQ_c \) concur at \( M \) on the cubic \( \mathcal{K}(O) \).

See Figure 3 and Remark following Construction 4 on the construction of tangents to \( \mathcal{K}(O) \).

2. Geometric properties of the Lemoine cubic

Proposition 1. The Lemoine cubic has the following geometric properties.

1. The two tangents at \( O \) are parallel to the asymptotes of the Jerabek hyperbola.

2. The tangent at \( H \) passes through the center \( X_{125} = [(b^2 - c^2)^2 S_A] \) of the Jerabek hyperbola. \(^3\)

3. The tangents at \( A, B, C \) concur at \( X_{184} = [a^4 S_A] \), the inverse of \( X_{125} \) in the Brocard circle.

4. The asymptotes are parallel to those of the orthocubic, i.e., the pivotal isogonal cubic with pivot \( H \).

5. The “third” intersections \( H_A, H_B, H_C \) of \( \mathcal{K}(O) \) and the altitudes lie on the circle with diameter \( OH \). \(^4\) The triangles \( A'B'C' \) and \( H_AH_BH_C \) are perspective at a point

\[
Z_1 = [a^4 S_A(a^4 + b^4 + c^4 - 2a^2(b^2 + c^2))]
\]

on the cubic. \(^5\)

6. The “third” intersections \( A'', B'', C'' \) of \( \mathcal{K}(O) \) and the sidelines of the medial triangle form a triangle perspective with \( H_AH_BH_C \) at a point

\[
Z_2 = \left[ \frac{a^4 S_A^2}{3a^4 - 2a^2(b^2 + c^2) - (b^2 - c^2)^2} \right]
\]

on the cubic. \(^6\)

7. \( \mathcal{K}(O) \) intersects the circumcircle of \( ABC \) at the vertices of the circumnormal triangle of \( ABC \). \(^7\)

\(^3\) This is also tangent to the Jerabek hyperbola at \( H \).

\(^4\) In other words, these are the projections of \( O \) on the altitudes. The coordinates of \( H_A \) are

\[
\left( \frac{2a^4 S_A}{a^2(b^2 + c^2) - (b^2 - c^2)^2} : S_C : S_B \right).
\]

\(^5\) \( Z_1 \) is the isogonal conjugate of \( X_{8427} \). It lies on a large number of lines, 13 using only triangle centers from [4], for example, \( X_2X_{54}, X_3X_{49}, X_4X_{110}, X_5X_{578}, X_{24}X_{52} \) and others.

\(^6\) This point \( Z_2 \) is not in the current edition of [4]. It lies on the lines \( X_3X_{64}, X_4X_{122} \) and \( X_{92}X_{253} \).

\(^7\) These are the points \( U, V, W \) on the circumcircle for which the lines \( UU^*, VV^*, WW^* \) (joining each point to its own isogonal conjugate) all pass through \( O \). As such, they are, together with the vertices, the intersections of the circumcircle and the McCay cubic, the isogonal cubic with pivot the circumcenter \( O \). See [3, p.166, §6.29].
We illustrate (1), (2), (3) in Figure 4, (4) in Figure 5, (5), (6) in Figure 6, and (7) in Figure 7 below.

Figure 4. The tangents to the Lemoine cubic at $O$ and the Jerabek hyperbola

Figure 5. The Lemoine cubic and the orthocubic have parallel asymptotes
Figure 6. The perspectors $Z_1$ and $Z_2$

Figure 7. The Lemoine cubic with the circumnormal triangle
3. The generalized Lemoine cubic

Let \( P \) be a point distinct from \( H \), not lying on any of the sidelines of triangle \( ABC \). Consider its pedal triangle \( P_aP_bP_c \). For every point \( M \) in the plane, let \( M_a = PP_a \cap AM \). Define \( M_b \) and \( M_c \) similarly. The locus of \( M \) such that the three points \( M_a, M_b, M_c \) are collinear on a line \( L_M \) is a cubic \( \mathcal{K}(P) \) called the generalized Lemoine cubic associated with \( P \). This cubic passes through \( A, B, C, H, P_a, P_b, P_c \), and \( P \) which is a node. Moreover, the line \( L_M \) envelopes the inscribed parabola with directrix the line \( HP \) and focus \( F \) the antipode (on the circumcircle) of the isogonal conjugate of the infinite point of the line \( HP \). The perspector \( S \) is the second intersection of the Steiner circum-ellipse with the line through \( F \) and the Steiner point \( X_{99} = \frac{1}{b^2-c^2} \).

With \( P = (p : q : r) \), the equation of \( \mathcal{K}(P) \) is

\[
\sum_{\text{cyclic}} x \left( r(c^2p + SBr)y^2 - q(b^2p + Scr)z^2 \right) + \left( \sum_{\text{cyclic}} a^2p(q - r) \right) xyz = 0.
\]

The two constructions in §1 can easily be adapted to this more general situation.

**Construction 3.** For any point \( Q \) on the trilinear polar of \( S \), the trilinear polar \( q \) of \( Q \) meets the lines \( PP_a, PP_b, PP_c \) at \( Q_a, Q_b, Q_c \) respectively. The lines \( AQ_a, BQ_b, CQ_c \) concur at \( M \) on the cubic \( \mathcal{K}(P) \).

**Construction 4.** For any point \( Q \) on the line \( HP \), the perpendicular bisector of \( FQ \) intersects the lines \( PP_a, PP_b, PP_c \) at \( Q_a, Q_b, Q_c \) respectively. The lines \( AQ_a, BQ_b, CQ_c \) concur at \( M \) on the cubic \( \mathcal{K}(P) \).

**Remark.** The tangent at \( M \) to \( \mathcal{K}(P) \) can be constructed as follows: the perpendicular at \( Q \) to the line \( HP \) intersects the perpendicular bisector of \( FQ \) at \( N \), which is the point of tangency of the line through \( Q_a, Q_b, Q_c \) with the parabola. The tangent at \( M \) to \( \mathcal{K}(P) \) is the tangent at \( M \) to the circum-conic through \( M \) and \( N \). Given a point \( M \) on the cubic, first construct \( M_a = AM \cap PP_a \) and \( M_b = BM \cap PP_b \), then \( Q \) the reflection of \( F \) in the line \( M_aM_b \), and finally apply the construction above.

Jean-Pierre Ehrmann has noticed that \( \mathcal{K}(P) \) can be seen as the locus of point \( M \) such that the circum-conic passing through \( M \) and the infinite point of the line \( PM \) is a rectangular hyperbola. This property gives another very simple construction of \( \mathcal{K}(P) \) or the construction of the “second” intersection of \( \mathcal{K}(P) \) and any line through \( P \).

**Construction 5.** A line \( \ell_P \) through \( P \) intersects \( BC \) at \( P_1 \). The parallel to \( \ell_P \) at \( A \) intersects \( HC \) at \( P_2 \). \( AB \) and \( P_1P_2 \) intersect at \( P_3 \). Finally, \( HP_3 \) intersects \( \ell_P \) at \( M \) on the cubic \( \mathcal{K}(P) \).

Most of the properties of the Lemoine cubic \( \mathcal{K}(O) \) also hold for \( \mathcal{K}(P) \) in general.

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\(^8\)Construction of \( F \) : draw the perpendicular at \( A \) to the line \( HP \) and reflect it about a bisector passing through \( A \). This line meets the circumcircle at \( A \) and \( F \).
**Proposition 2.** Let $\mathcal{K}(P)$ be the generalized Lemoine cubic.

1. The two tangents at $P$ are parallel to the asymptotes of the rectangular circum-hyperbola passing through $P$.

2. The tangent at $H$ to $\mathcal{K}(P)$ is the tangent at $H$ to the rectangular circum-hyperbola which is the isogonal image of the line $OF$. The asymptotes of this hyperbola are perpendicular and parallel to the line $HP$.

3. The tangents at $A, B, C$ concur if and only if $P$ lies on the Darboux cubic.\(^9\)

4. The asymptotes are parallel to those of the pivotal isogonal cubic with pivot the anticomplement of $P$.

5. The “third” intersections $H_A, H_B, H_C$ of $\mathcal{K}(P)$ with the altitudes are on the circle with diameter $HP$. The triangles $P_aP_bP_c$ and $H_AH_BH_C$ are perspective at a point on $\mathcal{K}(P)$.\(^10\)

6. The “third” intersections $A''$, $B''$, $C''$ of $\mathcal{K}(P)$ and the sidelines of $P_aP_bP_c$ form a triangle perspective with $H_AH_BH_C$ at a point on the cubic.

**Remarks.** (1) The tangent of $\mathcal{K}(P)$ at $H$ passes through the center of the rectangular hyperbola through $P$ if and only if $P$ lies on the isogonal non-pivotal cubic $\mathcal{K}_H$

\[ \sum_{\text{cyclic}} x (c^2y^2 + b^2z^2) - \Phi xyz = 0 \]

where

\[ \Phi = \frac{\sum_{\text{cyclic}} (2b^2c^2(a^4 + b^2c^2) - a^6(2b^2 + 2c^2 - a^2))}{4S_AS_BS_C}. \]

We shall study this cubic in §6.3 below.

(2) The polar conic of $P$ can be seen as a degenerate rectangular hyperbola. If $P \neq X_5$, the polar conic of a point is a rectangular hyperbola if and only if it lies on the line $PX_5$. From this, there is only one point (apart from $P$) on the curve whose polar conic is a rectangular hyperbola. Very obviously, the polar conic of $H$ is a rectangular hyperbola if and only if $P$ lies on the Euler line. If $P = X_5$, all the points in the plane have a polar conic which is a rectangular hyperbola. This very special situation is detailed in §4.2.

### 4. Special Lemoine cubics

4.1. $\mathcal{K}(P)$ with concuring asymptotes. The three asymptotes of $\mathcal{K}(P)$ are concurrent if and only if $P$ lies on the cubic $\mathcal{K}_{\text{conc}}$

\[ \sum_{\text{cyclic}} \left( S_B \left( c^2(a^2 + b^2) - (a^2 - b^2)^2 \right) y - S_C \left( b^2(a^2 + c^2) - (a^2 - c^2)^2 \right) z \right) x^2 \\
- 2(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)xyz = 0. \]

---

\(^9\)The Darboux cubic is the isogonal cubic with pivot the de Longchamps point $X_{20}$.

\(^10\)The coordinates of this point are $(p^2(-S_Ap + S_Bq + S_CR) + a^2pqr : \cdots : \cdots)$.
The three asymptotes of $K(P)$ are all real if and only if $P$ lies inside the Steiner deltoid $\mathcal{H}_3$.\textsuperscript{11} For example, the point $X_{76} = \left[ \frac{1}{a^2} \right]$ lies on the cubic $K_{\text{conc}}$ and inside the Steiner deltoid. The cubic $K(X_{76})$ has three real asymptotes concurring at a point on $X_5X_{76}$. See Figure 8. On the other hand, the de Longchamps point $X_{20}$ also lies on $K_{\text{conc}}$, but it is not always inside $\mathcal{H}_3$. See Figure 10. The three asymptotes of $K(X_{20})$, however, intersect at the real point $X_{376}$, the reflection of $G$ in $O$.

We shall study the cubic $K_{\text{conc}}$ in more detail in §6.1 below.

\textbf{Figure 8.} $K(X_{76})$ with three concurring asymptotes

4.2. $K(P)$ with asymptotes making 60° angles with one another. $K(P)$ has three real asymptotes making 60° angles with one another if and only if $P$ is the nine-point center $X_5$. See Figure 9. The asymptotes of $K(X_5)$ are parallel again to those of the McCay cubic and their point of concurrence is\textsuperscript{12}

$$Z_3 = \left[ a^2((b^2 - c^2)^2 - a^2(b^2 + c^2))(a^4 - 2a^2(b^2 + c^2) + b^4 - 5b^2c^2 + c^4) \right].$$

\textsuperscript{11}Cf. Cundy and Parry [1] have shown that for a pivotal isogonal cubic with pivot $P$, the three asymptotes are all real if and only if $P$ lies inside a certain “critical deltoid” which is the anticomplement of $\mathcal{H}_3$, or equivalently, the envelope of axes of inscribed parabolas.

\textsuperscript{12}$Z_3$ is not in the current edition of [4]. It is the common point of several lines, e.g. $X_5X_{51}$, $X_{373}X_{549}$ and $X_{511}X_{547}$.
4.3. **Generalized Lemoine isocubics.** $\mathcal{K}(P)$ is an isocubic if and only if the points $P_a, P_b, P_c$ are collinear. It follows that $P$ must lie on the circumcircle. The line through $P_a, P_b, P_c$ is the Simson line of $P$ and its trilinear pole $R$ is the root of the cubic. When $P$ traverses the circumcircle, $R$ traverses the Simson cubic. See [2]. The cubic $\mathcal{K}(P)$ is a conico-pivotal isocubic: for any point $M$ on the curve, its isoconjugate $M^*$ (under the isoconjugation with fixed point $P$) lies on the curve and the line $MM^*$ envelopes a conic. The points $M$ and $M^*$ are obtained from two points $Q$ and $Q'$ (see Construction 4) on the line $HP$ which are inverse with respect to the circle centered at $P$ going through $F$, focus of the parabola in §2. (see remark in §5 for more details)

5. **The construction of nodal cubics**

In §3, we have seen how to construct $\mathcal{K}(P)$ which is a special case of nodal cubic. More generally, we give a very simple construction valid for any nodal circum-cubic with a node at $P$, intersecting the sidelines again at any three points $P_a, P_b, P_c$. Let $R_a$ be the trilinear pole of the line passing through the points $AB \cap PP_b$ and $AC \cap PP_c$. Similarly define $R_b$ and $R_c$. These three points are collinear on a line $L$ which is the trilinear polar of a point $S$. For any point $Q$ on the line $L$, the trilinear polar $q$ of $Q$ meets $PR_a, PP_b, PP_c$ at $Q_a, Q_b, Q_c$ respectively. The lines $AQ_a, BQ_b, CQ_c$ concur at $M$ on the sought cubic and, as usual, $q$ envelopes the inscribed conic $\gamma$ with perspector $S$.

**Remarks.** (1) The tangents at $P$ to the cubic are those drawn from $P$ to $\gamma$. These tangents are
The Lemoine cubic

(i) real and distinct when $P$ is outside $\gamma$ and is a ”proper” node,
(ii) imaginary when $P$ is inside $\gamma$ and is an isolated point, or
(iii) identical when $P$ lies on $\gamma$ and is a cusp, the cuspidal tangent being the tangent at $P$ to $\gamma$.

It can be seen that this situation occurs if and only if $P$ lies on the cubic tangent at $Pa, Pb, Pc$ to the sidelines of $ABC$ and passing through the points $BC \cap P_aP_c$, $CA \cap P_cP_a$, $AB \cap P_aP_b$. In other words and generally speaking, there is no cuspidal circum-cubic with a cusp at $P$ passing through $Pa, Pb, Pc$.

(2) When $Pa, Pb, Pc$ are collinear on a line $\ell$, the cubic becomes a conico-pivotal isocubic invariant under isoconjugation with fixed point $P$: for any point $M$ on the curve, its isoconjugate $M^*$ lies on the curve and the line $MM^*$ envelopes the conic $\Gamma$ inscribed in the anticevian triangle of $P$ and in the triangle formed by the lines $AP_a, BP_b, CP_c$. The tangents at $P$ to the cubic are tangent to both conics $\gamma$ and $\Gamma$.

6. Some cubics related to $K(P)$

6.1. The cubic $K_{conc}$.

The circumcubic $K_{conc}$ considered in §4.1 above contains a large number of interesting points: the orthocenter $H$, the nine-point center $X_6$, the de Longchamps point $X_{20}, X_{70}$, the point

$$Z_4 = \left[ a^2S_A^2(a^2(b^2 + c^2) - (b^2 - c^2)^2) \right]$$

which is the anticomplement of $X_{389}$, the center of the Taylor circle. The cubic $K_{conc}$ also contains the traces of $X_{69}$ on the sidelines of $ABC$, the three cusps of the Steiner deltoid, and its contacts with the altitudes of triangle $ABC$. $Z$ is also the common point of the three lines each joining the trace of $X_{69}$ on a sideline of $ABC$ and the contact of the Steiner deltoid with the corresponding altitude. See Figure 10.

**Proposition 3.** The cubic $K_{conc}$ has the following properties.

1. The tangents at $A, B, C$ concur at $X_{53}$, the Lemoine point of the orthic triangle.
2. The tangent at $H$ is the line $HK$.
3. The tangent at $X_5$ is the Euler line of the orthic triangle, the tangential being the point $Z_4$. $Z_4$ is parallel to those of the McCay cubic and concur at a point $Z_5$.

$$Z_5 = \left[ a^2(a^2(b^2 + c^2) - (b^2 - c^2)^2)(2S_A^2 + b^2c^2) \right].$$

13 The point $Z_4$ is therefore the center of the Taylor circle of the antimedial triangle. It lies on the line $X_{4}X_{69}$.

14 The contact with the altitude $AH$ is the reflection of its trace on $BC$ about the midpoint of $AH$.

15 This line also contains $X_{51}, X_{52}$ and other points.

16 $Z_5$ is not in the current edition of [4]. It is the common point of quite a number of lines, e.g. $X_{3}X_{64}, X_{5}X_{51}, X_{113}X_{127}, X_{128}X_{130},$ and $X_{140}X_{185}$. The three asymptotes of the McCay cubic are concurrent at the centroid $G$. 

(5) $K_{\text{conc}}$ intersects the circumcircle at $A$, $B$, $C$ and three other points which are the antipodes of the points whose Simson lines pass through $X_{389}$.

We illustrate (1), (2), (3) in Figure 11, (4) in Figure 12, and (5) in Figure 13.
6.2. The isogonal image of $K(O)$. Under isogonal conjugation, $K(O)$ transforms into another nodal circum-cubic

$$
\sum_{cyclic} b^2 c^2 x (S_B y^2 - S_C z^2) + (a^2 - b^2)(b^2 - c^2)(c^2 - a^2)xyz = 0.
$$
The node is the orthocenter $H$. The cubic also passes through $O, X_8$ (Nagel point) and its extraversions, $X_{76}, X_{847} = Z_1^*$, and the traces of $X_{264} = \left[ \frac{1}{a^2 S_A} \right]$. The tangents at $H$ are parallel to the asymptotes of the Stammler rectangular hyperbola$^{17}$. The three asymptotes are concurrent at the midpoint of $GH,^{18}$ and are parallel to those of the McCay cubic.

![Figure 14. The Lemoine cubic and its isogonal](image)

This cubic was already known by J. R. Musselman [6] although its description is totally different. We find it again in [9] in a different context. Let $P$ be a point on the plane of triangle $ABC$, and $P_1, P_2, P_3$ the orthogonal projections of $P$ on the perpendicular bisectors of $BC, CA, AB$ respectively. The locus of $P$ such that the triangle $P_1P_2P_3$ is in perspective with $ABC$ is the Stammler hyperbola and the locus of the perspector is the cubic which is the isogonal transform of $K(O)$. See Figure 15.

$^{17}$The Stammler hyperbola is the rectangular hyperbola through the circumcenter, incenter, and the three excenters. Its asymptotes are parallel to the lines through $X_{110}$ and the two intersections of the Euler line and the circumcircle

$^{18}$This is $X_{381} = [a^2(a^2 + b^2 + c^2) - 2(b^2 - c^2)^2]$. 
6.3. The cubic $K_H$. Recall from Remark (1) following Proposition 2 that the tangent at $H$ to $K(P)$ passes through the center of the rectangular circum-hyperbola passing through $P$ if and only if $P$ lies on the cubic $K_H$. This is a non-pivotal isogonal circum-cubic with root at $G$. See Figure 14.

**Proposition 4.** The cubic $K_H$ has the following geometric properties.

1. $K_H$ passes through $A$, $B$, $C$, $O$, $H$, the three points $H_A$, $H_B$, $H_C$ and their isogonal conjugates $H_A^*$, $H_B^*$, $H_C^*$. \[19\]
2. The three real asymptotes are parallel to the sidelines of $ABC$.
3. The tangents of $K_H$ at $A$, $B$, $C$ are the sidelines of the tangential triangle. Hence, $K_H$ is tritangent to the circumcircle at the vertices $A$, $B$, $C$.
4. The tangent at $A$ (respectively $B$, $C$) and the asymptote parallel to $BC$ (respectively $CA$, $AB$) intersect at a point $\tilde{A}$ (respectively $\tilde{B}$, $\tilde{C}$) on $K_H$.
5. The three points $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are collinear on the perpendicular $L$ to the line $OK$ at the inverse of $X_{389}$ in the circumcircle. \[20\]

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19 The points $H_A$, $H_B$, $H_C$ are on the circle, diameter $OH$. See Proposition 1(5). Their isogonal conjugates are on the lines $OA$, $OB$, $OC$ respectively.

20 In other words, the line $L$ is the inversive image of the circle with diameter $OX_{389}$. Hence, $\tilde{A}$ is the common point of $L$ and the tangent at $A$ to the circumcircle, and the parallel through $\tilde{A}$ to $BC$ is an asymptote of $K_H$. 
(6) The isogonal conjugate of $\tilde{A}$ is the “third” intersection of $K_H$ with the parallel to $BC$ through $A$; similarly for the isogonal conjugates of $\tilde{B}$ and $\tilde{C}$.

(7) The third intersection with the Euler line, apart from $O$ and $H$, is the point $Z_6 = \left[ \frac{(b^2 - c^2)^2 + a^2(b^2 + c^2 - 2a^2)}{(b^2 - c^2)^2} S_A \right]$.

(8) The isogonal conjugate of $Z_6$ is the sixth intersection of $K_H$ with the Jerabek hyperbola.

We conclude with another interesting property of the cubic $K_H$. Recall that the polar circle of triangle $ABC$ is the unique circle with respect to which triangle $ABC$ is self-polar. This is in the coaxal system generated by the circumcircle and the nine-point circle. It has center $H$, radius $\rho$ given by

$$\rho^2 = 4R^2 - \frac{1}{2}(a^2 + b^2 + c^2),$$

and is real only when triangle $ABC$ is obtuse angled. Let $C$ be the concentric circle with radius $\frac{\rho}{\sqrt{2}}$.

**Proposition 5.** $K_H$ is the locus of point $M$ whose pedal circle is orthogonal to circle $C$.

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21This is not in [4]. It is the homothetic of $X_{402}$ (Gossard perspector) in the homothety with center $G$, ratio 4 or, equivalently, the anticomplement of the anticomplement of $X_{402}$. 

In fact, more generally, every non-pivotal isogonal cubic can be seen, in a unique way, as the locus of point $M$ such that the pedal circle of $M$ is orthogonal to a fixed circle, real or imaginary, proper or degenerate.

References


Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
*E-mail address: b.gibert@free.fr*