Triangle Centers Associated with the Malfatti Circles

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Abstract. Various formulae for the radii of the Malfatti circles of a triangle are presented. We also express the radii of the excircles in terms of the radii of the Malfatti circles, and give the coordinates of some interesting triangle centers associated with the Malfatti circles.

1. The radii of the Malfatti circles

The Malfatti circles of a triangle are the three circles inside the triangle, mutually tangent to each other, and each tangent to two sides of the triangle. See Figure 1. Given a triangle \(ABC\), let \(a, b, c\) denote the lengths of the sides \(BC, CA, AB\), \(s\) the semiperimeter, \(I\) the incenter, and \(r\) its inradius. The radii of the Malfatti circles of triangle \(ABC\) are given by

\[
\begin{align*}
    r_1 &= \frac{r}{2(s-a)} \left( s - r - (IB + IC - IA) \right), \\
    r_2 &= \frac{r}{2(s-b)} \left( s - r - (IC + IA - IB) \right), \\
    r_3 &= \frac{r}{2(s-c)} \left( s - r - (IA + IB - IC) \right).
\end{align*}
\]

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According to F.G.-M. [1, p.729], these results were given by Malfatti himself, and were published in [7] after his death. See also [6]. Another set of formulae give the same radii in terms of $a$, $b$, $c$ and $r$:

$$r_1 = \frac{(IB + r - (s - b))(IC + r - (s - c))}{2\left(IA + r - (s - a)\right)},$$
$$r_2 = \frac{(IC + r - (s - c))(IA + r - (s - a))}{2\left(IB + r - (s - b)\right)},$$
$$r_3 = \frac{(IA + r - (s - a))(IB + r - (s - b))}{2\left(IC + r - (s - c)\right)}.$$ (2)

These easily follow from (1) and the following formulae that express the radii $r_1$, $r_2$, $r_3$ in terms of $r$ and trigonometric functions:

$$r_1 = \frac{\left(1 + \tan \frac{B}{4}\right)\left(1 + \tan \frac{C}{4}\right)}{1 + \tan \frac{A}{4}} \cdot \frac{r}{2},$$
$$r_2 = \frac{\left(1 + \tan \frac{C}{4}\right)\left(1 + \tan \frac{A}{4}\right)}{1 + \tan \frac{B}{4}} \cdot \frac{r}{2},$$
$$r_3 = \frac{\left(1 + \tan \frac{A}{4}\right)\left(1 + \tan \frac{B}{4}\right)}{1 + \tan \frac{C}{4}} \cdot \frac{r}{2}.$$ (3)

These can be found in [10]. They can be used to obtain the following formula which is given in [2, pp.103–106]. See also [12].

$$\frac{2}{r} = \frac{1}{\sqrt{r_1 r_2}} + \frac{1}{\sqrt{r_2 r_3}} + \frac{1}{\sqrt{r_1 r_3}} - \sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}.$$ (4)

2. Exradii in terms of Malfatti radii

Antreas P. Hatzipolakis [3] asked for the exradii $r_a$, $r_b$, $r_c$ of triangle $ABC$ in terms of the Malfatti radii $r_1$, $r_2$, $r_3$ and the inradius $r$.

Proposition 1.

$$r_a - r_1 = \frac{\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}\right)},$$
$$r_b - r_2 = \frac{\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}\right)},$$
$$r_c - r_3 = \frac{\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}\right)}.$$ (5)
Proof. For convenience we write
\[ t_1 := \tan \frac{A}{4}, \quad t_2 := \tan \frac{B}{4}, \quad t_3 := \tan \frac{C}{4}. \]

Note that from \( \tan \left( \frac{A}{4} + \frac{B}{4} + \frac{C}{4} \right) = 1 \), we have
\[ 1 - t_1 - t_2 - t_3 - t_1 t_2 - t_2 t_3 - t_3 t_1 + t_1 t_2 t_3 = 0. \quad (6) \]

From (3) we obtain
\[ \frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}} = \frac{t_1}{1 + t_1} \cdot \frac{2}{r}, \]
\[ \frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}} = \frac{t_2}{1 + t_2} \cdot \frac{2}{r}, \]
\[ \frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}} = \frac{t_3}{1 + t_3} \cdot \frac{2}{r}. \quad (7) \]

For the exradius \( r_a \), we have
\[ r_a = \frac{s}{s-a} \cdot r = \cot \frac{B}{2} \cot \frac{C}{2} \cdot r = \frac{(1-t_2^2)(1-t_3^2)}{4a t_2 t_3} \cdot r. \]

It follows that
\[ r_a - r_1 = (1 + t_2)(1 + t_3) \cdot \frac{r}{2} \left( \frac{(1-t_2)(1-t_3)}{2t_2 t_3} - \frac{1}{1 + t_1} \right) \]
\[ = (1 + t_2)(1 + t_3) \cdot \frac{r}{2} \cdot \frac{(1+t_1)(1-t_2)(1-t_3) - 2t_2 t_3}{2t_2 t_3(1 + t_1)} \]
\[ = (1 + t_2)(1 + t_3) \cdot \frac{r}{2} \cdot \frac{2t_1}{2t_2 t_3(1 + t_1)} \quad \text{(from (6))} \]
\[ = \frac{t_1}{1 + t_1} \cdot \frac{1 + t_2}{t_2} \cdot \frac{1 + t_3}{t_3} \cdot \frac{r}{2}. \]

Now the result follows from (7). \( \square \)

Note that with the help of (4), the exradii \( r_a, r_b, r_c \) can be explicitly written in terms of the Malfatti radii \( r_1, r_2, r_3 \). We present another formula useful in the next sections in the organization of coordinates of triangle centers.

Proposition 2.
\[ \frac{1}{r_1} - \frac{1}{r_a} = \frac{a}{rs} \cdot \frac{(1 + \cos \frac{B}{2})(1 + \cos \frac{C}{2})}{1 + \cos \frac{A}{2}}. \]

3. Triangle centers associated with the Malfatti circles

Let \( A' \) be the point of tangency of the Malfatti circles \( C_2 \) and \( C_3 \). Similarly define \( B' \) and \( C' \). It is known ([4, p.97]) that triangle \( A'B'C' \) is perspective with \( ABC \) at the first Ajima-Malfatti point \( X_{179} \). See Figure 3. We work out the details here and construct a few more triangle centers associated with the Malfatti circles. In particular, we find two new triangle centers \( P_+ \) and \( P_- \) which divide the incenter \( I \) and the first Ajima-Malfatti point harmonically.
3.1. The centers of the Malfatti circles. We begin with the coordinates of the centers of the Malfatti circles.

Since $O_1$ divides the segment $AI_a$ in the ratio $AO_1 : O_1I_a = r_1 : r_a - r_1$, we have $\frac{O_1}{r_1} = \left( \frac{1}{r_1} - \frac{1}{r_a} \right) A + \frac{1}{r_a} : I_A$. With $r_a = \frac{r_a}{r_s}$ we rewrite the absolute barycentric coordinates of $O_1$, along with those of $O_2$ and $O_3$, as follows.

\[
\frac{O_1}{r_1} = \left( \frac{1}{r_1} - \frac{1}{r_a} \right) A + \frac{s - a}{r_s} \cdot I_a,
\frac{O_2}{r_2} = \left( \frac{1}{r_2} - \frac{1}{r_b} \right) B + \frac{s - b}{r_s} \cdot I_b,
\frac{O_3}{r_3} = \left( \frac{1}{r_3} - \frac{1}{r_c} \right) C + \frac{s - c}{r_s} \cdot I_c.
\]

From these expressions we have, in homogeneous barycentric coordinates,

\[
\frac{O_1}{r_1} = \left( \frac{2rs}{r_1} - \frac{1}{r_a} \right) - a : b : c,
\frac{O_2}{r_2} = \left( \frac{a}{2rs} - \frac{1}{r_b} \right) - b : c,
\frac{O_3}{r_3} = \left( \frac{a}{r_3} - \frac{1}{r_c} \right) - c.
\]
3.2. The triangle center $P_{-}$. It is clear that $O_1O_2O_3$ is perspective with $ABC$ at the incenter $(a : b : c)$. However, it also follows that if we consider
\[ A'' = BO_3 \cap CO_2, \quad B'' = CO_1 \cap AO_3, \quad C'' = AO_2 \cap BO_1, \]
then triangle $A''B''C''$ is perspective with $ABC$ at
\[
P_{-} = \left( 2rs \left( \frac{1}{r_1} - \frac{1}{r_a} \right) - a : 2rs \left( \frac{1}{r_2} - \frac{1}{r_b} \right) - b : 2rs \left( \frac{1}{r_3} - \frac{1}{r_c} \right) - c \right)
\]
\[= \left( \frac{1}{r_1} - \frac{1}{r_a} - \frac{a}{2rs} : \frac{1}{r_2} - \frac{1}{r_b} - \frac{b}{2rs} : \frac{1}{r_3} - \frac{1}{r_c} - \frac{c}{2rs} \right)
\]
\[= \left( a \left( \frac{(1 + \cos \frac{A}{2})(1 + \cos \frac{C}{2})}{1 + \cos \frac{A}{2}} - \frac{1}{2} \right) : \cdots : \cdots \right) \quad (9)\]

by Proposition 2. See Figure 2.

Remark. The point $P_{-}$ appears in [5] as the first Malfatti-Rabinowitz point $X_{1142}$.

3.3. The first Ajima-Malfatti point. For the points of tangency of the Malfatti circles, note that $A'$ divides $O_2O_3$ in the ratio $O_2A' : A'O_3 = r_2 : r_3$. We have, in absolute barycentric coordinates,
\[
\left( \frac{1}{r_2} + \frac{1}{r_3} \right) A' = \frac{O_2}{r_2} + \frac{O_3}{r_3} = \frac{a}{rs} \cdot A + \left( \frac{1}{r_2} - \frac{1}{r_b} \right) B + \left( \frac{1}{r_3} - \frac{1}{r_c} \right) C;
\]
similarly for $B'$ and $C'$. In homogeneous coordinates,
\[
A' = \left( a \left( \frac{1}{r_2} - \frac{1}{r_b} : \frac{1}{r_3} - \frac{1}{r_c} \right), \right.
\]
\[
B' = \left( \frac{1}{r_1} - \frac{1}{r_a} : \frac{b}{rs} : \frac{1}{r_3} - \frac{1}{r_c} \right), \quad (10)
\]
\[
C' = \left( \frac{1}{r_1} - \frac{1}{r_a} : \frac{1}{r_2} : \frac{c}{rs} \right).
\]

From these, it is clear that $A'B'C'$ is perspective with $ABC$ at
\[
P = \left( \frac{1}{r_1} - \frac{1}{r_a} : \frac{1}{r_2} - \frac{1}{r_b} : \frac{1}{r_3} - \frac{1}{r_c} \right)
\]
\[= \left( a(1 + \cos \frac{B}{2})(1 + \cos \frac{C}{2}) : b(1 + \cos \frac{C}{2})(1 + \cos \frac{A}{2}) \right)
\]
\[\quad : \frac{c(1 + \cos \frac{A}{2})(1 + \cos \frac{B}{2})}{1 + \cos \frac{C}{2}} \right)
\]
\[= \left( \frac{a}{(1 + \cos \frac{A}{2})^2} : \frac{b}{(1 + \cos \frac{B}{2})^2} : \frac{c}{(1 + \cos \frac{C}{2})^2} \right) \quad (11)\]
by Proposition 2. The point \( P \) appears as \( X_{179} \) in \([4, \text{p.}97]\), with trilinear coordinates

\[
\left( \sec^4 \frac{A}{4} : \sec^4 \frac{B}{4} : \sec^4 \frac{C}{4} \right)
\]

computed by Peter Yff, and is named the first Ajima-Malfatti point. See Figure 3.

![Figure 3](image)

3.4. The triangle center \( P_+ \). Note that the circle through \( A', B', C' \) is orthogonal to the Malfatti circles. It is the radical circle of the Malfatti circles, and is the incircle of \( O_1O_2O_3 \). The lines \( O_1A', O_2B', O_3C' \) are concurrent at the Gergonne point of triangle \( O_1O_2O_3 \). See Figure 4. As such, this is the point \( P_+ \) given by

\[
\left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right) P_+ = \frac{O_1}{r_1} + \frac{O_2}{r_2} + \frac{O_3}{r_3}
\]

\[
= \left( \frac{1}{r_1} - \frac{1}{r_a} \right) A + \frac{I_a}{r_a} + \left( \frac{1}{r_2} - \frac{1}{r_b} \right) B + \frac{I_b}{r_b} + \left( \frac{1}{r_3} - \frac{1}{r_c} \right) C + \frac{I_c}{r_c}
\]

\[
= \left( \frac{1}{r_1} - \frac{1}{r_a} \right) A + \left( \frac{1}{r_2} - \frac{1}{r_b} \right) B + \left( \frac{1}{r_3} - \frac{1}{r_c} \right) C + \frac{1}{r} (aA + bB + cC)
\]

\[
= \left( \frac{1}{r_1} - \frac{1}{r_a} + \frac{a}{2rs} \right) A + \left( \frac{1}{r_2} - \frac{1}{r_b} + \frac{b}{2rs} \right) B + \left( \frac{1}{r_3} - \frac{1}{r_c} + \frac{c}{2rs} \right) C.
\]
It follows that in homogeneous coordinates,
\[
P_+ = \left( \frac{1}{r_1} - \frac{1}{r_a} + \frac{a}{2rs}, \frac{1}{r_2} - \frac{1}{r_b} + \frac{b}{2rs}, \frac{1}{r_3} - \frac{1}{r_c} + \frac{c}{2rs} \right)
\]
\[
= \left( a \left( \frac{1 + \cos \frac{B}{2}}{1 + \cos \frac{C}{2}} \right) + \frac{1}{2} \right) : \cdots : \cdots \quad (12)
\]
by Proposition 2.

**Proposition 3.** The points \( P_+ \) and \( P_- \) divide the segment \( IP \) harmonically.

**Proof.** This follows from their coordinates given in (12), (9), and (11). \( \square \)

From the coordinates of \( P, P_+ \) and \( P_- \), it is easy to see that \( P_+ \) and \( P_- \) divide the segment \( IP \) harmonically.

3.5. **The triangle center** \( Q \). Let the Malfatti circle \( C_1 \) touch the sides \( CA \) and \( AB \) at \( Y_1 \) and \( Z_1 \) respectively. Likewise, let \( C_2 \) touch \( AB \) and \( BC \) at \( Z_2 \) and \( X_2 \), \( C_3 \) touch \( BC \) and \( CA \) at \( X_3 \) and \( Y_3 \) respectively. Denote by \( X, Y, Z \) the midpoints of the segments \( X_2X_3, Y_3Y_1, Z_1Z_2 \) respectively. Stanley Rabinowitz [9] asked if the lines \( AX, BY, CZ \) are concurrent. We answer this in the affirmative.

**Proposition 4.** The lines \( AX, BY, CZ \) are concurrent at a point \( Q \) with homogeneous barycentric coordinates
\[
\left( \tan \frac{A}{4} : \tan \frac{B}{4} : \tan \frac{C}{4} \right).
\]
Proof. In Figure 5, we have

\[ BX = \frac{1}{2} (a + BX_2 - X_3C) = \frac{1}{2} \left( a + \frac{r_2}{r} (s - b) - \frac{r_3}{r} (s - c) \right) = \frac{1}{2} (a + IB - IC) \quad \text{(from (1))} \]

\[ = \frac{1}{2} \left( 2R \sin A + \frac{r}{\sin \frac{B}{2}} - \frac{r}{\sin \frac{C}{2}} \right) = 4R \sin \frac{A}{2} \cos \frac{B}{4} \sin \frac{C}{4} \cos \frac{B + C}{4} \]

by making use of the formula

\[ r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \]

Similarly,

\[ XC = \frac{1}{2} (a - BX_2 + X_3C) = 4R \sin \frac{A}{2} \cos \frac{B}{4} \cos \frac{C}{4} \cos \frac{B + C}{4}. \]

It follows that

\[ \frac{BX}{XC} = \frac{\cos \frac{B}{4} \sin \frac{C}{4}}{\sin \frac{B}{4} \cos \frac{C}{4}} = \tan \frac{C}{4} \]

Likewise,

\[ \frac{CY}{YA} = \tan \frac{A}{4} \quad \text{and} \quad \frac{AZ}{ZB} = \tan \frac{C}{4}. \]
and it follows from Ceva’s theorem that $AX$, $BY$, $CZ$ are concurrent since
\[ \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1. \]

In fact, we can easily identify the homogeneous barycentric coordinates of the intersection $Q$ as given in (13) above since those of $X$, $Y$, $Z$ are
\[
\begin{align*}
X &= \left( 0 : \tan \frac{B}{4} : \tan \frac{C}{4} \right), \\
Y &= \left( \tan \frac{A}{4} : 0 : \tan \frac{C}{4} \right), \\
Z &= \left( \tan \frac{A}{4} : \tan \frac{B}{4} : 0 \right).
\end{align*}
\]

\[ \square \]

Remark. The coordinates of $Q$ can also be written as
\[
\left( \frac{\sin \frac{A}{2}}{1 + \cos \frac{A}{2}} : \frac{\sin \frac{B}{2}}{1 + \cos \frac{B}{2}} : \frac{\sin \frac{C}{2}}{1 + \cos \frac{C}{2}} \right)
\]
or
\[
\left( \frac{a}{(1 + \cos \frac{A}{2}) \cos \frac{B}{2}} : \frac{b}{(1 + \cos \frac{B}{2}) \cos \frac{C}{2}} : \frac{c}{(1 + \cos \frac{C}{2}) \cos \frac{A}{2}} \right).
\]

3.6. The radical center of the Malfatti circles. Note that the common tangent of $C_2$ and $C_3$ at $A'$ passes through $X$. This means that $A'X$ is perpendicular to $O_2O_3$ at $A'$. This line therefore passes through the incenter $I'$ of $O_1O_2O_3$. Now, the homogeneous coordinates of $A'$ and $X$ can be rewritten as
\[
\begin{align*}
A' &= \left( \frac{a}{(1 + \cos \frac{A}{2})(1 + \cos \frac{B}{2})(1 + \cos \frac{C}{2})} : \frac{b}{(1 + \cos \frac{B}{2})^2} : \frac{c}{(1 + \cos \frac{C}{2})^2} \right), \\
X &= \left( 0 : \frac{b}{(1 + \cos \frac{B}{2}) \cos \frac{B}{2}} : \frac{c}{(1 + \cos \frac{C}{2}) \cos \frac{C}{2}} \right).
\end{align*}
\]

It is easy to verify that these two points lie on the line
\[
\frac{(1 + \cos \frac{A}{2})(\cos \frac{B}{2} - \cos \frac{C}{2})}{a} x - \frac{(1 + \cos \frac{B}{2}) \cos \frac{B}{2}}{b} y + \frac{(1 + \cos \frac{C}{2}) \cos \frac{C}{2}}{c} z = 0,
\]
which also contains the point
\[
\left( \frac{a}{1 + \cos \frac{A}{2}} : \frac{b}{1 + \cos \frac{B}{2}} : \frac{c}{1 + \cos \frac{C}{2}} \right).
\]

Similar calculations show that the latter point also lies on the lines $BY$ and $C'Z$. It is therefore the incenter $I'$ of triangle $O_1O_2O_3$. See Figure 6. This point appears in [5] as $X_{183}$, the radical center of the Malfatti circles.
Remarks. (1) The line joining $Q$ and $I'$ has equation
\[
\frac{(1 + \cos \frac{A}{2})(\cos \frac{B}{2} - \cos \frac{C}{2})}{\sin \frac{A}{2}} x + \frac{(1 + \cos \frac{B}{2})(\cos \frac{C}{2} - \cos \frac{A}{2})}{\sin \frac{B}{2}} y + \frac{(1 + \cos \frac{C}{2})(\cos \frac{A}{2} - \cos \frac{B}{2})}{\sin \frac{C}{2}} z = 0.
\]
This line clearly contains the point $(\sin \frac{A}{2} : \sin \frac{B}{2} : \sin \frac{C}{2})$, which is the point $X_{174}$, the Yff center of congruence in [4, pp.94–95].

(2) According to [4], the triangle $A'B'C'$ in §3.3 is also perspective with the excentral triangle. This is because cevian triangles and anticevian triangles are always perspective. The perspector
\[
\left( a \left( (2 + \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2})^2 + \cos \frac{A}{2} (\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} - (2 + \cos \frac{A}{2})^2) \right) \right) : \cdots : \cdots
\]
is named the second Ajima-Malfatti point $X_{180}$. For the same reason, the triangle $XYZ$ in §3.5 is also perspective with the excentral triangle. The perspector is the point
\[
\left( a \left( -\cos \frac{A}{2} \left( 1 + \cos \frac{A}{2} \right) + \cos \frac{B}{2} \left( 1 + \cos \frac{B}{2} \right) + \cos \frac{C}{2} \left( 1 + \cos \frac{C}{2} \right) \right) \right) : \cdots : \cdots.
\]
This point and the triangle center $P_+$ apparently do not appear in the current edition of [5].

Editor’s endnote. The triangle center $Q$ in §3.5 appears in [5] as the second Malfatti-Rabinowitz point $X_{1143}$. Its coordinates given by the present editor [13] were not correct owing to a mistake in a sign in the calculations. In the notations of [13], if...
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\( \alpha, \beta, \gamma \) are such that

\[
\sin^2 \alpha = \frac{a}{s}, \quad \sin^2 \beta = \frac{b}{s}, \quad \sin^2 \gamma = \frac{c}{s},
\]

and \( \lambda = \frac{1}{2}(\alpha + \beta + \gamma) \), then the homogeneous barycentric coordinates of \( Q \) are

\[
(\cot(\lambda - \alpha) : \cot(\lambda - \beta) : \cot(\lambda - \gamma)).
\]

These are equivalent to those given in (13) in simpler form.

References


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