Harcourt’s Theorem

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Abstract. We give a proof of Harcourt’s theorem that if the signed distances from the vertices of a triangle of sides $a$, $b$, $c$ to a tangent of the incircle are $a_1$, $b_1$, $c_1$, then $aa_1 + bb_1 + cc_1$ is twice of the area of the triangle. We also show that there is a point on the circumconic with center $I$ whose distances to the sidelines of $ABC$ are precisely $a_1$, $b_1$, $c_1$. An application is given to the extangents triangle formed by the external common tangents of the excircles.

1. Harcourt’s Theorem

The following interesting theorem appears in F. G.-M.[1, p.750] as Harcourt’s theorem.

Theorem 1 (Harcourt). If the distances from the vertices $A$, $B$, $C$ to a tangent to the incircle of triangle $ABC$ are $a_1$, $b_1$, $c_1$ respectively, then the algebraic sum $aa_1 + bb_1 + cc_1$ is twice of the area of triangle $ABC$.

The distances are signed. Distances to a line from points on opposite sides are opposite in sign, while those from points on the same side have the same sign. For the tangent lines to the incircle, we stipulate that the distance from the incenter is positive. For example, in Figure 1, when the tangent line $\ell$ separates the vertex $A$ from $B$ and $C$, $a_1$ is negative while $b_1$ and $c_1$ are positive. With this sign convention, Harcourt’s theorem states that

$$aa_1 + bb_1 + cc_1 = 2\Delta, \quad (1)$$

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where \( \triangle \) is the area of triangle \( ABC \).

We give a simple proof of Harcourt’s theorem by making use of homogeneous barycentric coordinates with reference to triangle \( ABC \). First, we establish a fundamental formula.

**Proposition 2.** Let \( \ell \) be a line passing through a point \( P \) with homogeneous barycentric coordinates \((x : y : z)\). If the signed distances from the vertices \( A, B, C \) to a line \( \ell \) are \( d_1, d_2, d_3 \) respectively, then

\[
d_1x + d_2y + d_3z = 0. \tag{2}
\]

**Proof.** It is enough to consider the case when \( \ell \) separates \( A \) from \( B \) and \( C \). We take \( d_1 \) as negative, and \( d_2, d_3 \) positive. See Figure 2. If \( A' \) is the trace of \( P \) on the side line \( BC \), it is well known that

\[
\frac{AP}{PA'} = \frac{x}{y + z}.
\]

![Figure 2](image)

Since \( \frac{BA'}{A'C} = \frac{z}{y} \), the distance from \( A' \) to \( \ell \) is

\[
d_1' = \frac{yd_2 + zd_3}{y + z}.
\]

Since \( \frac{-d_1}{d_1} = \frac{AP}{PA} = \frac{y + z}{x} \), the equation (2) follows. \( \square \)

**Proof of Harcourt’s theorem.** We apply Proposition 2 to the line \( \ell \) through the incenter \( I = (a : b : c) \) parallel to the tangent. The signed distances from \( A, B, C \) to \( \ell \) are \( d_1 = a_1 - r, d_2 = a_2 - r, \) and \( d_3 = a_3 - r \). From these,

\[
aa_1 + bb_1 + cc_1 = a(d_1 + r) + b(d_2 + r) + c(d_3 + r)
\]

\[= (ad_1 + bd_2 + cd_3) + (a + b + c)r \]

\[= 2\Delta,
\]

since \( ad_1 + bd_2 + cd_3 = 0 \) by Proposition 2.
2. Harcourt’s theorem for the excircles

Harcourt’s theorem for the incircle and its proof above can be easily adapted to the excircles.

**Theorem 3.** If the distances from the vertices $A$, $B$, $C$ to a tangent to the $A$-excircle of triangle $ABC$ are $a_1$, $b_1$, $c_1$ respectively, then $-aa_1 + bb_1 + cc_1 = 2\triangle$. Analogous statements hold for the $B$- and $C$-excircles.

**Proof.** Apply Proposition 2 to the line $\ell$ through the excenter $I_a = (-a : b : c)$ parallel to the tangent. If the distances from $A$, $B$, $C$ to $\ell$ are $d_1$, $d_2$, $d_3$ respectively, then

$$-ad_1 + bd_2 + cd_3 = 0.$$  

Since $a_1 = d_1 + r_1$, $b_1 = d_2 + r_1$, $c_1 = d_3 + r_1$, where $r_1$ is the radius of the excircle, it easily follows that

$$-aa_1 + bb_1 + cc_1 = -a(d_1 + r_1) + b(d_2 + r_1) + c(d_3 + r_1)$$

$$= (-ad_1 + bd_2 + cd_3) + r_1(-a + b + c)$$

$$= r_1(-a + b + c)$$

$$= 2\triangle.$$  

Consider the external common tangents of the excircles of triangle $ABC$. Let $\ell_a$ be the external common tangent of the $B$- and $C$-excircles. Denote by $d_{a1}$, $d_{a2}$, $d_{a3}$ the distances from the $A$, $B$, $C$ to this line. Clearly, $d_{a1} = h_a$, the altitude on $BC$. Similarly define $\ell_b$, $\ell_c$ and the associated distances.
Theorem 4. \(d_a d_b d_c = d_a d_b d_c\).  

Proof. Applying Theorem 3 to the tangent \(\ell_a\) of the \(B\)-excircle (respectively the \(C\)-excircle), we have 
\[
\begin{align*}
ad_{a1} - bd_{a2} + cd_{a3} &= 2\triangle, \\
ad_{a1} + bd_{a2} - cd_{a3} &= 2\triangle.
\end{align*}
\]
From these it is clear that \(bd_{a2} = cd_{a3}\), and 
\[
d_{a2} = \frac{c}{b}.
\]
Similarly, 
\[
d_{b3} = \frac{a}{c} \quad \text{and} \quad d_{c1} = \frac{b}{a}.
\]
Combining these three equations we have \(d_a d_b d_c = d_a d_b d_c\). \(\square\)

It is clear that the perpendiculars from \(A\) to \(\ell_a\), being the reflection of the \(A\)-altitude, passes through the circumcenter; similarly for the perpendiculars from \(B\) to \(\ell_b\) and from \(C\) to \(\ell_c\).

Let \(X\) be the intersection of the perpendiculars from \(B\) to \(\ell_c\) and from \(C\) to \(\ell_b\). Note that \(OB\) and \(CX\) are parallel, so are \(OC\) and \(BX\). Since \(OB = OC\), it follows that \(OBCX\) is a rhombus, and \(BX = CX = R\), the circumradius
of triangle $ABC$. It also follows that $X$ is the reflection of $O$ in the side $BC$. Similarly, if $Y$ is the intersection of the perpendiculars from $C$ to $\ell_a$ and from $A$ to $\ell_c$, and $Z$ that of the perpendiculars from $A$ to $\ell_b$ and from $B$ to $\ell_a$, then $XYZ$ is the triangle of reflections of the circumcenter $O$. As such, it is oppositely congruent to $ABC$, and the center of homothety is the nine-point center of triangle $ABC$.

3. The circum-ellipse with center $I$

Consider a tangent $\mathcal{L}$ to the incircle at a point $P$. If the signed distances from the vertices $A, B, C$ to $\mathcal{L}$ are $a_1, b_1, c_1$, then by Harcourt’s theorem, there is a point $P^#$ whose signed distances to the sides $BC, CA, AB$ are precisely $a_1, b_1, c_1$. What is the locus of the point $P^#$ as $P$ traverses the incircle? By Proposition 2, the barycentric equation of $\mathcal{L}$ is

$$a_1 x + b_1 y + c_1 z = 0.$$ 

This means that the point with homogeneous barycentric coordinates $(a : b : c)$ is a point on the dual conic of the incircle, which is the circumconic with equation

$$(s - a)yz + (s - b)zx + (s - c)xy = 0. \quad (3)$$

The point $P^#$ in question has barycentric coordinates $(aa_1 : bb_1 : cc_1)$. Since $(a_1, b_1, c_1)$ satisfies (3), if we put $(x, y, z) = (aa_1, bb_1, cc_1)$, then

$$a(s - a)yz + b(s - b)zx + c(s - c)xy = 0.$$ 

Thus, the locus of $P^#$ is the circumconic with perspector $(a(s - a) : b(s - b) : c(s - c))$.\(^1\) It is an ellipse, and its center is, surprisingly, the incenter $I$.\(^2\) We denote this circum-ellipse by $\mathcal{C}_I$. See Figure 5.

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\(^1\)This is the Mittenpunkt, the point $X_9$ in [4]. It can be constructed as the intersection of the lines joining the excenters to the midpoints of the corresponding sides of triangle $ABC$.

\(^2\)In general, the center of the circumconic $pyz + qzx + rxy = 0$ is the point with homogeneous barycentric coordinates $(p(q + r - p) : q(r + p - q) : r(p + q - r))$. 

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Figure 5
Let $A_1, B_1, C_1$ be the antipodes of the points of tangency of the incircle with the sidelines. It is quite easy to see that $A_1^#, B_1^#, C_1^#$ are the antipodes of $A, B, C$ in the circum-ellipse $C_I$. Note that $A_1^# B_1^# C_1^#$ and $ABC$ are oppositely congruent at $I$. It follows from Steiner’s porism that if we denote the intersections of $L$ and this ellipse by $Q^#$ and $R^#$, then the lines $P^#Q^#$ and $P^#R^#$ are tangent to the incircle at $Q$ and $R$. This leads to the following construction of $P^#$.

**Construction.** If the tangent to the incircle at $P$ intersects the ellipse $C_I$ at two points, the second tangents from these points to the incircle intersect at $P^#$ on $C_I$.

If the point of tangency $P$ has coordinates $\left(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}\right)$, with $u+v+w=0$, then $P^#$ is the point $\left(\frac{a(s-a)}{u} : \frac{b(s-b)}{v} : \frac{c(s-c)}{w}\right)$. In particular, if $L$ is the common tangent of the incircle and the nine-point circle at the Feuerbach point, which has coordinates $((s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2)$, then $P^#$ is the point $\left(\frac{a}{b+c} : \frac{b}{a+c} : \frac{c}{a+b}\right)$. This is $X_{100}$ of [3, 4]. It is a point on the circumcircle, lying on the half line joining the Feuerbach point to the centroid of triangle $ABC$. See [3, Figure 3.12, p.82].

### 4. The extangents triangle

Consider the external common tangent $\ell_a$ of the excircles $(I_b)$ and $(I_c)$. Let $d_{a1}$, $d_{a2}$, $d_{a3}$ be the distances from $A$, $B$, $C$ to this line. We have shown that $\frac{d_{a1}}{d_{a2}} = \frac{b}{c}$. On the other hand, it is clear that $\frac{d_{a1}}{d_{a2}} = \frac{b}{b+c}$. See Figure 6. It follows that $d_{a1} : d_{a2} : d_{a3} = bc : c(b+c) : b(b+c)$.

By Proposition 2, the barycentric equation of $\ell_a$ is

$$bcx + c(b+c)y + b(b+c)z = 0.$$  

Similarly, the equations of $\ell_b$ and $\ell_c$ are

$$c(c+a)x + cay + a(c+a)z =0,$$

$$b(a+b)x + a(a+b)y + abz =0.$$  

These three external common tangents bound a triangle called the **extangents triangle** in [3]. The vertices are the points

$A' = (-a^2s : b(c+a)(s-c) : c(a+b)(s-b))$,

$B' = (a(b+c)(s-c) : -b^2s : c(a+b)(s-a))$,

$C' = (a(b+c)(s-b) : b(c+a)(s-a) : -c^2s)$.

Let $I'_a$ be the incenter of the reflection of triangle $ABC$ in $A$. It is clear that the distances from $A$ and $I'_a$ to $\ell_a$ are respectively $h_a$ and $r$. Since $A$ is the midpoint of $II'_a$, the distance from $I$ to $\ell_a$ is $2h_a - r$.

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3The trilinear coordinates of these vertices given in [3, p.162, §6.17] are not correct. The diagonal entries of the matrices should read $1 + \cos A$ etc. and $\frac{\cos a + \cos b + \cos c}{(s-a)\alpha + (s-b)\beta + (s-c)\gamma}$ etc. respectively.
Now consider the reflection of $I$ in $O$. We denote this point by $I'$. Since the distances from $I$ and $O$ to $\ell_a$ are respectively $2h_a - r$ and $R + h_a$, it follows that the distance from $I'$ to $\ell_a$ is $2(R + h_a) - (2h_a - r) = 2R + r$. For the same reason, the distances from $I'$ to $\ell_b$ and $\ell_c$ are also $2R + r$. From this we deduce the following interesting facts about the extangents triangle.

**Theorem 5.** The extangent triangle bounded by $\ell_a$, $\ell_b$, $\ell_c$

1. has incenter $I'$ and inradius $2R + r$;
2. is perspective with the excentral triangle at $I'$;
3. is homothetic to the tangential triangle at the internal center of similitude of the circumcircle and the incircle of triangle $ABC$, the ratio of the homothety being $\frac{2R + r}{R}$.

**Proof.** It is enough to locate the homothetic center in (3). This is the point which divides $I'O$ in the ratio $2R + r : -R$, i.e.,

$$
\frac{(2R + r)O - R(2O - I)}{R + r} = \frac{r \cdot O + R \cdot I}{R + r},
$$

the internal center of similitude of the circumcircle and incircle of triangle $ABC$.\footnote{This point appears as $X_{55}$ in [4].}

□

**Remarks.** (1) The statement that the extangents triangle has inradius $2R + r$ can also be found in [2, Problem 2.5.4].

(2) Since the excentral triangle has circumcenter $I'$ and circumradius $2R$, it follows that the excenters and the incenters of the reflections of triangle $ABC$ in $A$, $B$, $C$ are concyclic. It is well known that since $ABC$ is the orthic triangle of the

\footnote{This point appears as $X_{40}$ in [4].}

\footnote{This point appears as $X_{55}$ in [4].}
excentral triangle, the circumcircle of $ABC$ is the nine-point circle of the excentral triangle.

(3) If the incircle of the extangents triangle touches its sides at $X, Y, Z$ respectively, then triangle $XYZ$ is homothetic to $ABC$, again at the internal center of similitude of the circumcircle and the incircle.

(4) More generally, the reflections of the traces of a point $P$ in the respective sides of the excentral triangle are points on the sidelines of the extangents triangle. They form a triangle perspective with $ABC$ at the isogonal conjugate of $P$. For example, the reflections of the points of tangency of the excircles (traces of the Nagel point $(s - a : s - b : s - c)$) form a triangle with perspector $\left(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}\right)$, the external center of similitude of the circumcircle and the incircle.\footnote{These are the reflections of the traces of the Gergonne point in the respective sides of the excentral triangle.}

References


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\footnote{This point appears as $X_{56}$ in [4].}