Extreme Areas of Triangles in Poncelet’s Closure Theorem

Mirko Radić

Abstract. Among the triangles with the same incircle and circumcircle, we determine the ones with maximum and minimum areas. These are also the ones with maximum and minimum perimeters and sums of altitudes.

Given two circles $C_1$ and $C_2$ of radii $r$ and $R$ whose centers are at a distance $d$ apart satisfying Euler’s relation

$$R^2 - d^2 = 2Rr,$$

by Poncelet’s closure theorem, for every point $A_1$ on the circle $C_2$, there is a triangle $A_1A_2A_3$ with incircle $C_1$ and circumcircle $C_2$. In this article we determine those triangles with extreme areas, perimeters, and sum of altitudes.

Denote by $t_m$ and $t_M$ respectively the lengths of the shortest and longest tangents that can be drawn from $C_2$ to $C_1$. These are given by

$$t_m = \sqrt{(R - d)^2 - r^2}, \quad t_M = \sqrt{(R + d)^2 - r^2}.$$

We shall use the following result given in [2, Theorem 2.2]. Let $t_1$ be any given length satisfying

$$t_m \leq t_1 \leq t_M.$$
and let $t_2$ and $t_3$ be given by

$$
t_2 = \frac{2Rrt_1 + \sqrt{D}}{r^2 + t_1^2}, \quad t_3 = \frac{2Rrt_1 - \sqrt{D}}{r^2 + t_1^2},
$$

where

$$
D = 4R^2r^2t_1^2 - r^2(r^2 + t_1^2)(4Rr + r^2 + t_1^2).
$$

Then there is a triangle $A_1A_2A_3$ with incircle $C_1$ and circumcircle $C_2$ with side lengths

$$
a_i = |A_iA_{i+1}| = t_i + t_{i+1}, \quad i = 1, 2, 3.
$$

Here, the indices are taken modulo 3. It is easy to check that

$$(t_1 + t_2 + t_3)r^2 = t_1t_2t_3,
$$

and that these are necessary and sufficient for $C_1$ and $C_2$ to be the incircle and circumcircle of triangle $A_1A_2A_3$.

Denote by $J(t_1)$ the area of triangle $A_1A_2A_3$. Thus,

$$
J(t_1) = r(t_1 + t_2 + t_3).
$$

Note that $D = 0$ when $t_1 = t_m$ or $t_1 = t_M$. In these cases,

$$
t_2 = t_3 = \begin{cases} 
\frac{2Rrt_m}{r^2 + t_m^2}, & \text{if } t_1 = t_m, \\
\frac{2Rrt_M}{r^2 + t_M^2}, & \text{if } t_1 = t_M.
\end{cases}
$$

For convenience, we shall write

$$
\hat{t}_m = \frac{2Rrt_m}{r^2 + t_m^2} \quad \text{and} \quad \hat{t}_M = \frac{2Rrt_M}{r^2 + t_M^2}.
$$

**Theorem 1.** $J(t_1)$ is maximum when $t_1 = t_M$ and minimum when $t_1 = t_m$. In other words, $J(t_m) \leq J(t_1) \leq J(t_M)$ for $t_m \leq t_1 \leq t_M$.

**Proof.** It follows from (6) and (4) that

$$
J(t_1) = r\left(t_1 + \frac{4Rrt_1}{r^2 + t_1^2}\right).
$$

From $\frac{d}{dt_1}J(t_1) = 0$, we obtain the equation

$$
t_1^4 - 2(2Rr - r^2)t_1^2 + 4Rr^3 + r^4 = 0,
$$

and

$$
t_1^2 = 2Rr - r^2 \pm 2r\sqrt{R^2 - 2Rr} = 2Rr - r^2 \pm 2rd.
$$

Since $4R^2r^2 = (R^2 - d^2)^2$, we have
Extreme areas of triangles in Poncelet’s closure theorem

\[ 2Rr - r^2 + 2rd - \hat{t}_m^2 \]
\[ = 2Rr - r^2 + 2rd - \frac{(R + d)^2((R - d)^2 - r^2)}{(R - d)^2} \]
\[ = \frac{(R - d)^2(2Rr - r^2 + 2rd) - (R + d)^2((R - d)^2 - r^2)}{(R - d)^2} \]
\[ = \frac{((R + d)^2 - (R - d)^2)r^2 + 2r(R + d)(R - d)^2 - (R^2 - d^2)^2}{(R - d)^2} \]
\[ = \frac{4Rdr^2 + 2r(R - d)(2Rr) - (2Rr)^2}{(R - d)^2} \]
\[ = 0. \]

Similarly, \( 2Rr - r^2 - 2rd - \hat{t}_M^2 = 0 \). It follows that \( \frac{d}{dt} J(t_1) = 0 \) for \( t_1 = \hat{t}_m, \hat{t}_M \). The maximum of \( J \) occurs at \( t_1 = t_M \) and \( \hat{t}_M \) while the minimum occurs at \( t_1 = t_m \) and \( \hat{t}_m \).

![Figure 2](image)

The triangle determined by \( \hat{t}_m \) (respectively \( \hat{t}_M \)) is exactly the one determined by \( t_m \) (respectively \( t_M \)).

We conclude with an interesting corollary. Let \( h_1, h_2, h_3 \) be the altitudes of the triangle \( A_1A_2A_3 \). Since
\[2R(h_1 + h_2 + h_3) = a_1a_2 + a_2a_3 + a_3a_1 = (t_1 + t_2 + t_3)^2 + 4Rr + r^2,\]

the following are equivalent:

- the triangle has maximum (respectively minimum) area,
- the triangle has maximum (respectively minimum) perimeter,
- the triangle has maximum (respectively minimum) sum of altitudes.

It follows that these are precisely the two triangles determined by \(t_M\) and \(t_m\).

References


Mirko Radić: Department of Mathematics, Faculty of Philosophy, University of Rijeka, 51000 Rijeka, Omladinska 14, Croatia

*E-mail address*: mradic@pefri.hr